

SELFDUAL EINSTEIN METRICS AND CONFORMAL SUBMERSIONS

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ABSTRACT. Weyl derivatives, Weyl-Lie derivatives and conformal submersions are defined, then used to generalize the Jones-Tod correspondence between selfdual 4-manifolds with symmetry and Einstein-Weyl 3-manifolds with an abelian monopole. In this generalization, the conformal symmetry is replaced by a particular kind of conformal submersion with one dimensional fibres. Special cases are studied in which the conformal submersion is holomorphic, affine, or projective. All scalar-flat Kähler metrics with such a holomorphic conformal submersion, and all four dimensional hypercomplex structures with a compatible Einstein metric, are obtained from solutions of the resulting “affine monopole equations”. The “projective monopole equations” encompass Hitchin’s twistorial construction of selfdual Einstein metrics from three dimensional Einstein-Weyl spaces, and lead to an explicit formula for carrying out this construction directly. Examples include new selfdual Einstein metrics depending explicitly on an arbitrary holomorphic function of one variable or an arbitrary axially symmetric harmonic function. The former generically have no continuous symmetries.

INTRODUCTION

The aims of this paper are threefold: firstly, to advertise the notion of a Weyl derivative both as a simple, but useful, tool in differential geometry, and also as an object of study in its own right; secondly, to apply this tool to the theory of conformal submersions, with particular attention to the case of selfdual conformal 4-manifolds; and thirdly to give explicit constructions of selfdual Einstein metrics. The key discovery is a class of conformal submersions with one dimensional fibres which admit a holomorphic interpretation on the twistor space. This class includes the conformal submersions generated by conformal vector fields and provides a natural setting for a generalized Jones-Tod correspondence [15].

A Weyl derivative on a manifold M is nothing more than a covariant derivative on a real line bundle naturally associated to the differential geometry of M . Weyl derivatives can be used to define Lie derivatives along foliations with one dimensional leaves, generalizing the usual Lie derivative along a vector field. They also occur naturally in the geometry of conformal submersions. These two situations have in common the conformal submersions with one dimensional fibres, to which most of this paper is devoted. I focus in particular on the case that the total space is a selfdual 4-manifold M and define the notion of a selfdual conformal submersion. In Theorem I (4.6), the base B of such a submersion is shown to be not just conformal, but Einstein-Weyl, generalizing the Jones-Tod correspondence to a

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context which also includes Hitchin's construction of selfdual Einstein metrics with Einstein-Weyl conformal infinity [13].

The central part of the paper deals with special cases of this construction. In Theorem II (4.11), shear-free geodesic congruences on B are shown to correspond to antiselfdual complex structures on M which are invariant with respect to the conformal submersion (i.e., the submersion is holomorphic), and the hypercomplex structures and scalar-flat Kähler metrics arising in this way are identified (generalizing [7, 11]). In Theorem III (4.13), an antiselfdual complex structure is constructed from a generic selfdual conformal submersion on a selfdual Einstein-Weyl 4-manifold—for a selfdual Einstein metric with a Killing field, this reduces to the construction of Tod [22].

The notion of an affine conformal submersion is defined, characterized, and shown, in Theorem IV (5.2), to provide a method for constructing selfdual spaces from coupled linear differential equations, which will be called *affine monopole equations*. In Theorem V (5.4), I show that the generalized Jones-Tod constructions for scalar-flat Kähler and hyperKähler metrics arise in this way—this includes LeBrun's construction of scalar flat metrics with Killing vector fields [17] and the construction of such metrics with homothetic vector fields [11, 7]. The same ideas are applied to projective conformal submersions in Theorem VI (6.3), giving *projective monopole equations*. This time the differential equations are nonlinear: they are the $SL(2, \mathbb{R})$ Einstein-Weyl Bogomolny equations.

In the rest of the paper, I study conformal submersions on Einstein-Weyl spaces, especially selfdual Einstein and locally hypercomplex 4-manifolds. In Theorem VII (7.2), Hitchin's version of LeBrun's \mathcal{H} -space construction of selfdual Einstein metrics is characterized amongst selfdual conformal submersions. In Theorem VIII (8.2), a larger class of selfdual conformal submersions is obtained, from pairs of compatible selfdual Einstein-Weyl structures. In Theorem IX (9.3), selfdual Einstein metrics with compatible hypercomplex structures are all found as affine conformal submersions over hyperCR Einstein-Weyl spaces. This special case of the Hitchin-LeBrun construction yields new selfdual Einstein metrics with no continuous symmetries. Finally, in Theorem X (10.1), an explicit formula for applying the Hitchin-LeBrun construction to any Einstein-Weyl space is given.

The paper is organized as follows. In section 1, Weyl derivatives and Weyl-Lie derivatives are introduced. Although it is possible to present some of the later results without this formalism, the proofs are simpler and more natural when one makes systematic use of the affine space of Weyl derivatives and the product rule for Weyl-Lie derivatives. Indeed, many of the results of this paper would have been impossible to find (for the author at least) without the geometric guidance provided by working in a gauge-independent way. In order to familiarize the reader with this language, I have presented a few simple applications of Weyl derivatives, and discussed carefully the Weyl-Lie derivative on natural bundles. The key formula from this section is the Weyl-Lie derivative of a torsion-free covariant derivative on a natural bundle. In section 2, after recalling basic facts from conformal geometry, I present another arrow in the Weyl geometer's quiver: the linearized Koszul formula.

In the third section, the notion of a conformal submersion is defined, but most of the local properties are studied within the more general framework of conformal almost product structures. I prove a simple proposition which shows that there is a canonically defined Weyl derivative in this setting, which will be called the *minimal Weyl derivative*. The main interest, however, is in conformal submersions with one dimensional fibres, which may be analyzed locally using the congruences and Weyl-Lie derivatives of section 1. I focus on this case for the second half of section 3 and characterize basic objects using the minimal Weyl-Lie derivative.

The generalized Jones-Tod correspondence (Theorem I) is established in section 4. In an earlier version of this paper [4], my proof followed closely the proof of the “classical” Jones-Tod correspondence given in [6, 7]. However, Paul Gauduchon has recently obtained a cleaner proof by exploiting more thoroughly the isomorphism between the bundle of antiselfdual 2-forms on the conformal 4-manifold M and the pullback of the tangent bundle of the 3-dimensional quotient B . This approach also has the advantage of integrating nicely with the fact that invariant antiselfdual complex structures on M correspond to shear-free geodesic congruences on B [7], and so it is Gauduchon’s proof that I follow here, adapted to the context of conformal submersions.

This generalized Jones-Tod construction is difficult to apply, because the generalized monopole equation and the defining equation for conformal submersions are both nonlinear. This difficulty is partially overcome in section 5 by studying a special case in which the construction linearizes: affine conformal submersions. In this case M is an affine bundle over B such that the nonlinear connection and relative length scale induced by the conformal structure are affine. A characterization of such submersions is presented and the affine monopole equations are obtained. These linear differential equations give constructions of scalar-flat Kähler and hyperKähler metrics extending work of LeBrun [17] and Pedersen and myself [7]. I illustrate this with some examples, taken from [8]. The following section extends these ideas to projective conformal submersions. The projective monopole equations are not linear, but they will play a crucial role in the final section.

Several of the results in this paper were motivated by twistor theory. I explain this in section 7, where I also prove that the Hitchin-LeBrun construction, defined twistorially in [13], really is a special case of the generalized Jones-Tod correspondence, and this special case is characterized. The Hitchin-LeBrun construction, although simple from a twistor point of view, has always been notoriously difficult to carry out explicitly due to a lack of a direct construction: the known examples are, to the best of my knowledge, those of [16, 13, 20, 8]. The main result of the final portion of the paper is Theorem X, which reduces the Hitchin-LeBrun construction to an explicit formula for a selfdual Einstein metric in terms of an arbitrary Einstein-Weyl structure. Firstly, though, a special case, Theorem IX, is established using affine conformal submersions. A key tool here is the observation that two compatible Einstein-Weyl structures on a conformal manifold define a conformal submersion, and in four dimensions, this submersion is selfdual if the conformal structure is. This is proven in section 8. Theorem IX then characterizes all selfdual Einstein metrics admitting compatible hypercomplex structures.

Explicit examples, with no continuous symmetries, are given. They depend on an arbitrary holomorphic function of one variable.

In the final section, a projective gauge transformation is applied to the affine monopoles of Theorem IX. The result is a canonical solution of the projective monopole equations which makes sense on any Einstein-Weyl space. In Theorem X, I show that the explicit metric given by the induced projective conformal submersion is Einstein. The resulting direct method for carrying out the Hitchin-LeBrun construction is illustrated by a family of selfdual Einstein metrics depending on an arbitrary axially symmetry harmonic function on \mathbb{R}^3 .

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This paper has had a long evolution (for a primitive version see [4]) and conversations with many others have also been influential. I am grateful to Florin Belgun and André Moroianu for the idea to write the generalized monopole equations as evolution equations. I would also especially like to thank: Vestislav Apostolov, for discussing (bi-)hypercomplex structures and Einstein metrics with me; Tammo Diemer, for pointing out, among other things, the Koszul formula in Weyl geometry; Nigel Hitchin for suggesting that I should look for affine structures compatible with conformal submersions; and Henrik Pedersen, both for initiating my work on Weyl structures, conformal submersions and selfdual Einstein metrics, and also for many discussions relating to the Hitchin-LeBrun construction.

Snapshots from the affine space of Weyl derivatives in sections 8 and 9 were produced with the aid of Paul Taylor's Commutative Diagrams package.

1. WEYL DERIVATIVES AND WEYL-LIE DERIVATIVES

If V is a real n -dimensional vector space and w any real number, then the oriented one dimensional linear space $L^w = L^w(V)$ carrying the representation $A \mapsto |\det A|^{w/n}$ of $\mathrm{GL}(V)$ is called the space of *densities of weight w* or *w -densities*. It can be constructed canonically as the space of maps $\rho: (\Lambda^n V) \setminus 0 \rightarrow \mathbb{R}$ such that $\rho(\lambda\omega) = |\lambda|^{-w/n}\rho(\omega)$ for all $\lambda \in \mathbb{R}^\times$ and $\omega \in (\Lambda^n V) \setminus 0$.

The same construction can be carried out pointwise on any vector bundle E to give, for each $w \in \mathbb{R}$, the oriented real line bundle L_E^w whose fibre at x is $L^w(E_x)$. Applying this to the tangent bundle gives the following definition.

1.1. Definition. Suppose M is any n -manifold. Then the *density line bundle* $L^w = L_{TM}^w$ of M is defined to be the bundle whose fibre at $x \in M$ is $L^w(T_x M)$.

Equivalently it is the associated bundle $\mathrm{GL}(M) \times_{\mathrm{GL}(n)} L^w(n)$ where $\mathrm{GL}(M)$ is the frame bundle of M and $L^w(n)$ is the space of w -densities of \mathbb{R}^n .

The density bundles are oriented (hence trivializable) real line bundles, but there is no preferred trivialization. Sections of $L = L^1$ may be thought of as scalar fields with dimensions of length. This geometric dimensional analysis may also be applied to tensors: the tensor bundle $L^w \otimes (TM)^j \otimes (T^*M)^k$ (and any subbundle, quotient bundle, element or section) is said to have *weight* $w + j - k$, or dimensions of $[\text{length}]^{w+j-k}$. Note that $L^{w_1} \otimes L^{w_2}$ is canonically isomorphic to $L^{w_1+w_2}$ and L^0 is the trivial bundle. When tensoring a vector bundle with some L^w (or any real line bundle), I shall often omit the tensor product sign. Note also that an orientation of M may be viewed as a unit section of $L^n \Lambda^n T^*M$, defining an isomorphism between L^{-n} and $\Lambda^n T^*M$. A nonvanishing (usually positive) section of L^1 (or L^w for $w \neq 0$) is called a *length scale* or *gauge* (of weight w).

1.2. Definition. A *Weyl derivative* is a covariant derivative D on L^1 . It induces covariant derivatives on L^w for all w . The curvature of D is a real 2-form F^D called the *Faraday curvature*.

If $F^D = 0$ then D is said to be *closed*. There are then local length scales μ with $D\mu = 0$. If such a length scale exists globally then D is said to be *exact*. Conversely, a length scale μ induces an exact Weyl derivative D^μ such that $D^\mu \mu = 0$. Note that $D^{c\mu} = D^\mu$ for any constant $c \neq 0$. Weyl derivatives form an affine space modelled on the linear space of 1-forms, while closed and exact Weyl derivatives are affine subspaces modelled on the linear spaces of closed and exact 1-forms respectively.

A *gauge transformation* on M is a positive function e^f which rescales a gauge $\mu \in C^\infty(M, L^w)$ to give $e^{wf}\mu$. It acts on Weyl derivatives by $e^f \cdot D = e^f \circ D \circ e^{-f} = D - df$, so that $e^f \cdot D^\mu = D^{e^f \mu}$ for $\mu \in C^\infty(M, L^1)$. If D is any Weyl derivative, then $D = D^\mu + \omega^\mu$ for the 1-form $\omega^\mu = \mu^{-1}D\mu$, and consequently, $\omega^{e^f \mu} = \omega^\mu + df$.

On an oriented manifold, Weyl derivatives may be viewed as a generalization of volume forms, since the exact Weyl derivatives correspond to volume forms up to constant multiples. For instance, the divergence of a vector field X with respect to a volume form vol is defined by $\mathcal{L}_X \text{vol} = (\text{div } X) \text{vol}$. In fact the divergence is naturally defined on vector field densities $C^\infty(M, L^{-n} T^*M)$, and by twisting by a Weyl derivative D on L^n one can define $\text{div}^D X$ for vector fields.

For another example, let $\Omega \in L^2 \Lambda^2 T^*M$ be a weightless 2-form on M^{2m} such that Ω^m is an orientation. Then Ω is nondegenerate and one would like to find a length scale μ such that $\mu^{-2}\Omega$ is symplectic. If $2m > 2$ this may not be possible.

1.3. Proposition. (cf. [18]) *Let M be an n -manifold ($n = 2m > 2$) and let $\Omega \in L^2 \Lambda^2 T^*M$ be nondegenerate. Then there is a unique Weyl derivative D such that $d^D \Omega$ is tracefree with respect to Ω , in the sense that $\sum d^D \Omega(e_i, e'_i, \cdot) = 0$, where e_i, e'_i are frames for $L^{-1} T^*M$ with $\Omega(e_i, e'_j) = \delta_{ij}$.*

Proof. Pick any Weyl derivative D^0 and set $D = D^0 + \gamma$ for some 1-form γ . Then $d^D \Omega = d^{D^0} \Omega + 2\gamma \wedge \Omega$ and so the traces differ by

$$2(\gamma \wedge \Omega)(e_i, e'_i, \cdot) = 2\gamma(e_i)\Omega(e'_i, \cdot) + 2\gamma(e'_i)\Omega(\cdot, e_i) + 2\gamma\Omega(e_i, e'_i) = 2(n-2)\gamma.$$

Since $n > 2$ it follows that there is a unique γ such that $d^D\Omega$ is tracefree. \square

There are therefore two local obstructions to finding μ with $d(\mu^{-2}\Omega) = 0$, namely $d^D\Omega$ and F^D [18]. In four dimensions $d^D\Omega$ automatically vanishes. In general if $d^D\Omega = 0$ then $F^D \wedge \Omega = 0$. In six or more dimensions this implies $F^D = 0$, so D is closed, but it need not be exact. In four dimensions, however, D need not even be closed. This construction is of particular interest in Hermitian geometry [23].

There is also a version of this in contact geometry. A *contact structure* on M is codimension one subbundle \mathcal{H} of TM which is maximally nonintegrable, in the sense that the Frobenius tensor $\Omega_{\mathcal{H}}: \Lambda^2\mathcal{H} \rightarrow TM/\mathcal{H}$ is nondegenerate. In this context, one defines a (slightly generalized) Weyl derivative to be a covariant derivative on the real line bundle TM/\mathcal{H} . The following result is then obtained.

1.4. Proposition. *Let M, \mathcal{H} be a contact manifold. Then there is a bijection between complementary subspaces to \mathcal{H} in TM and Weyl derivatives on TM/\mathcal{H} such that the horizontal part of F^D is tracefree with respect to $\Omega_{\mathcal{H}}$.*

Proof. Let $\eta_{\mathcal{H}}: TM \rightarrow TM/\mathcal{H}$ be the twisted contact 1-form whose kernel defines \mathcal{H} . Then given any Weyl derivative D , one can define $d^D\eta_{\mathcal{H}}$ and if $D = D^0 + \gamma$ then $d^D\eta_{\mathcal{H}} = d^{D^0}\eta_{\mathcal{H}} + \gamma \wedge \eta_{\mathcal{H}}$. Note that $d^D\eta_{\mathcal{H}}|_{\mathcal{H}}$ is well defined, being equal to $\Omega_{\mathcal{H}}$. Since $\Omega_{\mathcal{H}}$ is nondegenerate, given D there is a unique complementary subspace (spanned by ξ , say) such that $d^D\eta_{\mathcal{H}}(\xi, \cdot) = 0$ and such a complementary subspace fixes D up to 1-forms γ with $\eta_{\mathcal{H}} \wedge \gamma = 0$.

Now note that if $D = D^0 + \mu^{-1}\eta_{\mathcal{H}}$ for a section μ of TM/\mathcal{H} , then $F^D|_{\mathcal{H}} = F^{D^0}|_{\mathcal{H}} + \mu^{-1}\Omega_{\mathcal{H}}$ and consequently, D may be found uniquely with tracefree horizontal Faraday curvature. \square

This generalizes the fact that a contact form (which corresponds to a section of TM/\mathcal{H}) defines a Reeb vector field complementary to the contact distribution. Note also that it is perhaps more natural to work with “horizontal” covariant derivatives in this context, i.e., covariant differentiation is only defined along directions in \mathcal{H} . Then the condition on F^D can be ignored, and the bijection becomes affine.

Much of the rest of the paper is concerned with the dual situation of one dimensional subbundles of TM (the complementary subspaces arising above being an example). The integral manifolds of such a distribution define a foliation of M with one dimensional leaves. This will be viewed as an unparameterized version of a vector field by thinking of such a subbundle as an inclusion $\xi: \mathcal{V} \rightarrow TM$ of a real line bundle \mathcal{V} , and hence as a “twisted” vector field. Sections of \mathcal{V} correspond to vector fields tangent to the foliation. It is therefore natural to consider covariant derivatives on \mathcal{V} , which will again be referred to as Weyl derivatives. I will also use the following terminology from relativity.

1.5. Definition. A *congruence* on a manifold M is a nonvanishing section ξ of $\mathcal{V}^{-1}TM$ for some oriented real line bundle \mathcal{V} . It defines an oriented one dimensional subbundle of TM and hence a foliation with oriented one dimensional leaves.

No use will be made of the orientation of \mathcal{V} in this section.

1.6. Proposition. *Let E be a vector bundle associated to the frame bundle of M with induced representation ρ of $\mathfrak{gl}(TM)$. Then for a congruence ξ , a Weyl derivative D (on \mathcal{V}), and a section s of E , the formula*

$$\mu^{-1}\mathcal{L}_{\mu\xi}s + \rho(\mu^{-1}D\mu \otimes \xi)s$$

is independent of the choice of a nonvanishing section μ of \mathcal{V} , and will be called the Weyl-Lie derivative $\mathcal{L}_\xi^D s$ of s along ξ .

Proof. This follows from the fact that for any vector field X and function f , $\mathcal{L}_f X s = f\mathcal{L}_X s - \rho(df \otimes X)s$. \square

Note that $\mathcal{L}_\xi^D s$ is a section of $\mathcal{V}^{-1}E$. If D is exact, then trivializing \mathcal{V} by a parallel section gives back the usual Lie derivative. The dependence of the Weyl-Lie derivative on D is clearly given by $\mathcal{L}_\xi^{D+\gamma}s = \mathcal{L}_\xi^D s + \rho(\gamma \otimes \xi)s$. In particular, for functions (ρ trivial), $\mathcal{L}_\xi^D f = df(\xi)$, which is a section of \mathcal{V}^{-1} independent of D .

In order to compute the Weyl-Lie derivative, it is convenient to find a formula in terms of a torsion-free connection inducing covariant derivatives ∇ on any associated bundle E . In the case of the usual Lie derivative, $\mathcal{L}_K X = \nabla_K X - \nabla_X K$ for vector fields X , and so $\mathcal{L}_K s = \nabla_K s - \rho(\nabla K)s$ on sections of E . This readily yields the following generalization to Weyl-Lie derivatives:

$$(1.1) \quad \mathcal{L}_\xi^D s = \nabla_\xi s - \rho((D \otimes \nabla)\xi)s,$$

where $(D \otimes \nabla)\xi$ denotes the *twisted* or *tensor sum* covariant derivative of ξ as a section of $\mathcal{V}^{-1}TM = \mathcal{V}^{-1} \otimes TM$. More generally $D \otimes \nabla$ will denote the twisted covariant derivative on $\mathcal{V}^{-1}E$. Formula (1.1) immediately gives the following.

1.7. Proposition. *Let ξ be a congruence and D be a Weyl derivative (on \mathcal{V}). Then the Weyl-Lie derivative may be computed in terms of an arbitrary torsion-free covariant derivative ∇ as follows:*

- For a w -density μ , $\mathcal{L}_\xi^D \mu = \nabla_\xi \mu - \frac{w}{n}(\operatorname{div}^{D \otimes \nabla} \xi)\mu$.
- For a vector field X , $\mathcal{L}_\xi^D X = \nabla_\xi X - (D \otimes \nabla)_X \xi$.
- For a 1-form α , $\mathcal{L}_\xi^D \alpha = \nabla_\xi \alpha + \alpha((D \otimes \nabla)\xi) = d\alpha(\xi, \cdot) + D(\alpha(\xi))$.
- For a k -form α , $\mathcal{L}_\xi^D \alpha = \iota_\xi d\alpha + d^D(\iota_\xi \alpha)$.

Here $\operatorname{div}^{D \otimes \nabla} \xi$ denotes the trace of $(D \otimes \nabla)\xi$, i.e., the divergence has been twisted by D on \mathcal{V}^{-1} and ∇ on L^n .

The above treatment only deals with the Weyl-Lie derivative for zero and first order geometric objects (functions and sections of bundles associated to the first order frame bundle). It may be extended to differential operators (higher order geometric objects) using the product rule, but because the Weyl-Lie derivative of a section of E is not a section of E , the differential operators have to be twisted by D . Consequently, some natural differential operators may have nonzero Weyl-Lie derivative. In particular $(\mathcal{L}_\xi^D d)\alpha = \mathcal{L}_\xi^D(d\alpha) - d^D(\mathcal{L}_\xi^D \alpha) = F^D \wedge \iota_\xi \alpha$, i.e., the Weyl-Lie derivative only commutes with exterior differentiation on functions in general. Similarly, for the Lie bracket, $\mathcal{L}_\xi^D[\cdot, \cdot](X, Y) = -F^D(X, Y)\xi$. Of course, this is compatible with the definition of d in terms of $[\cdot, \cdot]$.

If ∇ is any torsion-free covariant derivative then its Weyl-Lie derivative $\mathcal{L}_\xi^D \nabla = \mathcal{L}_\xi^D \circ \nabla - (D \otimes \nabla) \circ \mathcal{L}_\xi^D$ evaluates to

$$(1.2) \quad (\mathcal{L}_\xi^D \nabla)_X = R_{\xi, X}^\nabla + (D \otimes \nabla)_X((D \otimes \nabla)\xi) \in \mathcal{V}^{-1} \mathfrak{gl}(TM).$$

This acts on an associated bundle E via the corresponding representation ρ of $\mathfrak{gl}(TM)$. The vanishing of $\mathcal{L}_\xi^D \nabla$ defines a notion of invariance, along ξ , of ∇ on E . Also note that $\nabla_\xi - \mathcal{L}_\xi^D = \rho((D \otimes \nabla)\xi)$ is a kind of Higgs field. If the Higgs field vanishes, I will say ∇ is horizontal, for reasons that will become clear later. In particular if ∇ is invariant and horizontal on E , then $\rho(R_{\xi, X}^\nabla) = 0$.

2. CONFORMAL GEOMETRY

In the previous section, the term “Weyl derivative” was sometimes applied in a generalized sense, when the oriented real line bundle was not necessarily L^1 . Such a distinction disappears when one introduces a conformal structure.

2.1. Definition. A *conformal structure* \mathfrak{c} on M is a metric on $L^{-1}TM$. Since the densities of $L_x^{-1}T_xM$ are canonically trivial, it makes sense to require that this metric is *normalized* in the sense that it has determinant one.

This is precisely what is needed in order to identify any oriented one dimensional subbundle or quotient bundle of TM with L^1 . In particular if M is conformal and ξ is a congruence, then there is a unique oriented isomorphism between \mathcal{V} and L^1 such that $\xi \in C^\infty(M, L^{-1}TM)$ is a weightless unit vector field. Thus a congruence will be viewed as an injective linear map $\xi: L^1 \rightarrow TM$ and \mathcal{V} will denote its image.

A conformal structure may be viewed as a fibrewise inner product on TM with values in L^2 : compatible Riemannian metrics therefore correspond bijectively to length scales. As in the previous section, it is natural to replace length scales with Weyl derivatives. Denoting the conformal inner product of vector fields by $\langle X, Y \rangle \in C^\infty(M, L^2)$, the Koszul formula for the Levi-Civita connection generalizes.

2.2. The fundamental theorem of conformal geometry. [25] *On a conformal manifold M there is an affine bijection between Weyl derivatives and torsion-free connections on TM preserving the conformal structure. More explicitly, the torsion-free connection on TM is determined by the Koszul formula*

$$\begin{aligned} 2\langle D_X Y, Z \rangle = & D_X \langle Y, Z \rangle + D_Y \langle X, Z \rangle - D_Z \langle X, Y \rangle \\ & + \langle [X, Y], Z \rangle - \langle [X, Z], Y \rangle - \langle [Y, Z], X \rangle. \end{aligned}$$

The corresponding linear map sends a 1-form γ to the $\mathfrak{co}(TM)$ -valued 1-form Γ defined by $\Gamma_X = \llbracket \gamma, X \rrbracket = \gamma(X)id + \gamma \Delta X$, where $(\gamma \Delta X)(Y) = \gamma(Y)X - \langle X, Y \rangle \gamma$.

2.3. Remark. Here, and elsewhere, I freely identify a 1-form γ with a vector field of weight -1 using the natural isomorphism $\sharp: T^*M \rightarrow L^{-2}TM$ given by the conformal structure. More generally, vector fields of any weight are identified with 1-forms of the same weight. Similarly a skew linear map J on TM (of any weight) corresponds to a 2-form Ω_J (of the same weight) via $J(X) = \sharp(\iota_X \Omega_J)$; this identifies $\gamma \Delta X$ with $\gamma \wedge \langle X, . \rangle$. It will sometimes, but not always, be helpful to maintain

a notational distinction between a skew endomorphism and the corresponding 2-form. The bracket $\llbracket \cdot, \cdot \rrbracket$ is part of an algebraic Lie bracket on $TM \oplus \mathfrak{co}(TM) \oplus T^*M$, and the same notation will be used for the commutator bracket on $\mathfrak{co}(TM)$.

The curvature R^D of D , as a $\mathfrak{co}(TM)$ -valued 2-form, decomposes as follows:

$$(2.1) \quad R_{X,Y}^D = W_{X,Y} - \llbracket r^D(X), Y \rrbracket + \llbracket r^D(Y), X \rrbracket.$$

Here W is the *Weyl curvature* of the conformal structure, an $\mathfrak{so}(TM)$ -valued 2-form, and r^D is a covector valued 1-form, the *normalized Ricci tensor* of D .

If ξ is a congruence, then the Weyl-Lie derivative of D on L^1 reduces to

$$(2.2) \quad (\mathcal{L}_\xi D)_X^{L^1} \mu = F^D(\xi, X)\mu + \frac{1}{n} D_X^\xi (\operatorname{div}^{D^\xi \otimes D} \xi) \mu,$$

where D^ξ is the Weyl derivative used to define \mathcal{L}_ξ . Linearizing the Koszul formula gives a formula for the Weyl-Lie derivative on other bundles:

$$(\mathcal{L}_\xi D)_X = D_X(\mathcal{L}_\xi c) + \operatorname{alt}(D(\mathcal{L}_\xi c)(X)) + \llbracket \mathcal{L}_\xi D^{L^1}, X \rrbracket + \frac{1}{2} \xi \triangle F^\xi(X) + \frac{1}{2} \langle \xi, X \rangle F^\xi,$$

where F^ξ is the Faraday curvature of D^ξ : these F^ξ terms come from the Weyl-Lie derivative of the Lie bracket. All the terms apart from the first belong to the conformal Lie algebra $\mathfrak{co}(TM)$. This reflects the fact that if $\mathcal{L}_\xi c = 0$, the Weyl-Lie derivative preserves conformal subbundles of natural vector bundles and so extends to any bundle associated to the conformal frame bundle.

3. CONFORMAL SUBMERSIONS

3.1. Definition. Let $\pi: M \rightarrow B$ be a smooth surjective map between conformal manifolds and let the *horizontal bundle* \mathcal{H} be the orthogonal complement to the vertical bundle \mathcal{V} of π in TM . Then π will be called a *conformal submersion* iff for all $x \in M$, $d\pi_x|_{\mathcal{H}_x}$ is a nonzero conformal linear map.

It is not at all necessary to restrict attention to submersions. The base could, for instance, be an orbifold, or be replaced altogether by the horizontal geometry of a foliation. However, since I am primarily interested in the local geometry, I shall usually take the base to be a manifold. A bundle \mathcal{H} complementary to \mathcal{V} is often called a (*nonlinear*) *connection* on π : it is equivalently determined by a projection $\eta: TM \rightarrow \mathcal{V}$, the *connection 1-form*.

3.2. Proposition. *If $\pi: M \rightarrow B$ is a submersion onto a conformal manifold B , then conformal structures on M making π into a conformal submersion correspond bijectively to triples $(\mathcal{H}, c^\mathcal{V}, w)$, where \mathcal{H} is a connection on π , $c^\mathcal{V}$ is a conformal structure on the fibres, and $w: \pi^* L_{TB}^1 \cong L_{\mathcal{H}}^1 \rightarrow L_{\mathcal{V}}^1$ is a (positive) isomorphism.*

Proof. $d\pi: \mathcal{H} \rightarrow \pi^* TB$ is certainly an isomorphism, so $L_{\mathcal{H}}^2 \cong \pi^* L_{TB}^2$ and the conformal structure on \mathcal{H} is obtained by pullback. Combining this with $c^\mathcal{V}$ gives an $L_{\mathcal{V}}^2$ valued inner product $c = c^\mathcal{V} \oplus w^2 c^\mathcal{H}$, which in turn determines an isomorphism between $L_{\mathcal{V}}$ and L_{TM} such that c becomes a conformal structure on M . \square

The final ingredient w in this construction will be called a *relative length scale*, since it allows vertical and horizontal lengths to be compared. The freedom to vary w generalizes the so called “canonical variation” of a Riemannian submersion, in which the fibre metric is rescaled, while the base metric remains constant.

A natural generalization of conformal submersions, following Gray [12], is a *conformal almost product structure*. On a conformal manifold M , this is a nontrivial orthogonal direct sum decomposition $TM = \mathcal{V} \oplus_{\perp} \mathcal{H}$, i.e., \mathcal{V} and \mathcal{H} are nontrivial subbundles of TM and are orthogonal complements with respect to the conformal structure. Although the roles of \mathcal{V} and \mathcal{H} are interchangeable, I will call the corresponding tangent directions vertical and horizontal.

Given any Weyl derivative D , observe that the vertical component $(D_X Y)^{\mathcal{V}}$ for $X, Y \in \mathcal{H}$ is tensorial in Y and so defines a tensor in $\mathcal{H}^* \otimes \mathcal{H}^* \otimes \mathcal{V}$. I will write $(D_X Y)^{\mathcal{V}} = \Pi_{\mathcal{H}}^D(X, Y) + \frac{1}{2}\Omega_{\mathcal{H}}(X, Y)$, where $\Pi_{\mathcal{H}}^D$ is symmetric and $\Omega_{\mathcal{H}}$ is skew, and extend these fundamental forms by zero to $T^*M \otimes T^*M \otimes TM$. Similarly the horizontal component $(D_U V)^{\mathcal{H}}$ for $U, V \in \mathcal{V}$ defines tensors $\Pi_{\mathcal{V}}^D$ and $\Omega_{\mathcal{V}}$ in $\mathcal{V}^* \otimes \mathcal{V}^* \otimes \mathcal{H} \leq T^*M \otimes T^*M \otimes TM$. Since D is torsion-free, $\Omega_{\mathcal{H}}$ and $\Omega_{\mathcal{V}}$ are the Frobenius tensors of the distributions \mathcal{H} and \mathcal{V} , which vanish iff the distributions are tangent to foliations. On the other hand, the fundamental forms $\Pi_{\mathcal{H}}^D$ and $\Pi_{\mathcal{V}}^D$ do depend on D and can be used to find a distinguished Weyl derivative D^0 .

3.3. Proposition. *Suppose M is conformal with a conformal almost product structure $TM = \mathcal{V} \oplus_{\perp} \mathcal{H}$. Then there is a unique Weyl derivative D^0 such that \mathcal{V} and \mathcal{H} are minimal, in the sense that the fundamental forms, denoted $\Pi_{\mathcal{H}}^0$ and $\Pi_{\mathcal{V}}^0$, are tracefree. It may be computed from an arbitrary Weyl derivative D via the formula:*

$$D^0 = D - \frac{\text{tr } \Pi_{\mathcal{H}}^D}{\dim \mathcal{H}} - \frac{\text{tr } \Pi_{\mathcal{V}}^D}{\dim \mathcal{V}}.$$

Proof. Observe that if $\tilde{D} = D + \gamma$ then $\langle \text{tr } \Pi_{\mathcal{H}}^{\tilde{D}}, U \rangle = \langle \text{tr } \Pi_{\mathcal{H}}^D, U \rangle + (\dim \mathcal{H})\gamma(U)$ for all $U \in \mathcal{V}$, and similarly for $\Pi_{\mathcal{V}}$. This shows that the formula for D^0 is independent of D . Substituting $D = D^0$ shows that $\Pi_{\mathcal{H}}^0$ and $\Pi_{\mathcal{V}}^0$ are tracefree. \square

I shall refer to D^0 as the *minimal Weyl derivative* of ξ ; this usage is consonant both with minimal submanifolds and minimal coupling. The minimal Weyl derivative need not be closed: I shall write F^0 for its Faraday curvature. Also I denote the minimal Weyl-Lie derivative along ξ by \mathcal{L}_{ξ}^0 .

3.4. Remark. The curvature of D^0 on TM can be related to the curvatures of horizontal and vertical connections on \mathcal{H} and \mathcal{V} . One defines a horizontal connection on \mathcal{H} by $D_X^{\mathcal{H}} Y = (D_X^0 Y)^{\mathcal{H}}$ for $X, Y \in \mathcal{H}$; similarly $D_U^{\mathcal{V}} V = (D_U^0 V)^{\mathcal{V}}$ for $U, V \in \mathcal{V}$. The (modified) curvature of $D^{\mathcal{H}}$ is defined by

$$R_{X,Y}^{\mathcal{H}} Z = D_X^{\mathcal{H}} D_Y^{\mathcal{H}} Z - D_Y^{\mathcal{H}} D_X^{\mathcal{H}} Z - D_{[X,Y]^{\mathcal{H}}}^{\mathcal{H}} Z - [[X, Y]^{\mathcal{V}}, Z]^{\mathcal{H}},$$

where $X, Y, Z \in \mathcal{H}$. The definition of $R^{\mathcal{V}}$ is analogous, and O’Neill-type formulae [19] relating R^{D^0} to $R^{\mathcal{H}}$ and $R^{\mathcal{V}}$ follow directly as in [12].

Some of the properties of the minimal Weyl derivative may be elucidated by comparing it with partial Lie derivatives. If X is a horizontal vector field and U is

vertical, then $[U, X]^{\mathcal{H}}$ is tensorial in U , which defines a partial covariant derivative on \mathcal{H} in vertical directions. This extends naturally to horizontal forms in $\Lambda^k \mathcal{H}^*$ and to densities in $L^1_{\mathcal{H}}$. One says a horizontal vector field, form, or density is invariant if its partial covariant derivative along \mathcal{V} vanishes; if \mathcal{V} is a tangent to the fibres of a submersion, this means the horizontal vector field, form, or density is basic.

Similarly, there is a partial covariant derivative on \mathcal{V} , $\Lambda^k \mathcal{V}^*$ and $L^1_{\mathcal{V}}$ in horizontal directions. Since the conformal structure identifies $L^1_{\mathcal{H}}$ with $L^1_{\mathcal{V}}$, putting these together gives a Weyl derivative. In order to verify that this is the minimal Weyl derivative defined above, introduce an arbitrary Weyl connection D so that $[U, X]^{\mathcal{H}} = (D_U X)^{\mathcal{H}} + (D_U X)^{\mathcal{V}}$ and the second term is tensorial: since $\langle D_U X, V \rangle = -\langle X, D_U V \rangle$ this tensor is essentially $\Pi^D_{\mathcal{V}} + \frac{1}{2}\Omega_{\mathcal{V}}$. Therefore the vertical partial connections on $L^1_{\mathcal{H}}$ induced by $[U, X]^{\mathcal{H}}$ and $(D_U X)^{\mathcal{H}}$ agree if and only if $\text{tr } \Pi^D_{\mathcal{V}} = 0$. Similarly the horizontal partial connections on $L^1_{\mathcal{V}}$ induced by $[X, U]^{\mathcal{V}}$ and $(D_X U)^{\mathcal{V}}$ agree if and only if $\text{tr } \Pi^D_{\mathcal{H}} = 0$.

According to this discussion, the following definition for densities is compatible with the identification of L^1 with $L^1_{\mathcal{H}}$ and $L^1_{\mathcal{V}}$.

3.5. Definition. A density $\mu \in C^\infty(M, L^1)$ on a conformal manifold M with a conformal almost product structure $(\mathcal{V}, \mathcal{H})$ is *invariant along \mathcal{V}* iff $D_U^0 \mu = 0$ for all vertical U , and *invariant along \mathcal{H}* iff $D_X^0 \mu = 0$ for all horizontal X .

I specialize now to the case that \mathcal{V} is one dimensional and oriented. Then the positively oriented weightless unit vector field ξ spanning $L^{-1}\mathcal{V} \leq L^{-1}TM$ is a congruence and the minimal Weyl derivative D^0 is characterized by $D_\xi^0 \xi = 0$ and $\text{tr } D^0 \xi = 0$ (note that $\langle D^0 \xi, \xi \rangle = 0$ since ξ has unit length). The formula for computing D^0 reduces to $D^0 = D - \frac{1}{n-1}(\text{div}^D \xi)\xi + (d^D \xi)(\xi, \cdot)$. Also note that $\Omega_{\mathcal{V}}$ and $\Pi^0_{\mathcal{V}}$ both vanish, so Ω and Π^0 will denote the fundamental forms of \mathcal{H} .

3.6. Proposition. Let ξ be a congruence with minimal Weyl derivative D^0 and let D be an arbitrary Weyl derivative.

- (i) $\frac{1}{2}\mathcal{L}_\xi^0 \mathbf{c} = \text{sym}_0(D^0 \otimes D)\xi = \text{sym } D^0 \xi = \Pi^0$.
- (ii) D^0 is exact iff $\xi = K/|K|$ for some (nonvanishing) vector field K which is divergence-free with respect to the metric $g = |K|^{-2}\mathbf{c}$.
- (iii) If D^0 is exact and $\mathcal{L}_\xi^0 \mathbf{c} = 0$ then K is a conformal vector field, and hence is a Killing field of the metric $g = |K|^{-2}\mathbf{c}$.

Conversely if K is a nonvanishing conformal vector field then $\xi = K/|K|$ is a congruence with $\mathcal{L}_\xi^0 \mathbf{c} = 0$ and $D^0|K| = 0$.

Proof. For the first part, note that for any vector fields X, Y ,

$$\begin{aligned} (\mathcal{L}_\xi^0 \mathbf{c})(X, Y) &= \mathcal{L}_\xi^0 \langle X, Y \rangle - \langle \mathcal{L}_\xi^0 X, Y \rangle - \langle X, \mathcal{L}_\xi^0 Y \rangle \\ &= D_\xi \langle X, Y \rangle - \frac{2}{n}(\text{div}^{D^0 \otimes D} \xi) \langle X, Y \rangle \\ &\quad - \langle D_\xi X - (D^0 \otimes D)_X \xi, Y \rangle - \langle X, D_\xi Y - (D^0 \otimes D)_Y \xi \rangle \\ &= \langle (D^0 \otimes D)_X \xi, Y \rangle + \langle X, (D^0 \otimes D)_Y \xi \rangle - \frac{2}{n}(\text{div}^{D^0 \otimes D} \xi) \langle X, Y \rangle. \end{aligned}$$

This is $2(\text{sym}_0(D^0 \otimes D)\xi)(X, Y) = 2(\text{sym } D^0\xi)(X, Y)$, since D was arbitrary. If either X or Y is parallel to ξ , $(\text{sym } D^0\xi)(X, Y)$ vanishes automatically, because $D^0\xi = 0 = \langle D^0\xi, \xi \rangle$. On the other hand, if X and Y are orthogonal to ξ , it is equal to $\frac{1}{2}\langle \xi, D_X^0 Y + D_Y^0 X \rangle = \langle \xi, \Pi^0(X, Y) \rangle$.

For the second and third parts, observe that an exact D^0 preserves a length scale μ . Then $K = \mu\xi$ and D^0 is the Levi-Civita connection of $g = \mu^{-2}\mathbf{c}$. \square

I shall now assume that the congruence ξ is tangent to the one dimensional fibres of a submersion over a manifold B ; this is always true locally. A horizontal vector field, form or density is then basic if it is invariant in the sense above. The Weyl-Lie derivative \mathcal{L}_ξ^0 provides an efficient way to characterize such basic objects.

3.7. Proposition. *Let ξ be a congruence with minimal Weyl derivative D^0 generating a submersion of M over B . Then a horizontal vector field X is basic iff $\mathcal{L}_\xi^0 X = 0$. Similarly a horizontal form α is basic iff $\mathcal{L}_\xi^0 \alpha = 0$. Finally a density μ is basic iff $\mathcal{L}_\xi^0 \mu = 0$.*

Proof. It suffices to check that $\mathcal{L}_\xi^0 X = 0$ is equivalent to $[U, X]$ being vertical for all vertical U . If $U = \lambda\xi$ then away from the zero set of λ , $D^\lambda = D^0 + \gamma$ for some 1-form γ and so $[U, X] = \mathcal{L}_{\lambda\xi} X = \lambda\mathcal{L}_\xi^{D^\lambda} X = \lambda\mathcal{L}_\xi^0 X + \gamma(X)\lambda\xi$. Hence $\mathcal{L}_\xi^0 X$ is vertical if and only if $[U, X]$ is vertical for all vertical U . For the vertical component observe that $\langle \xi, \mathcal{L}_\xi^0 X \rangle = -\mathcal{L}_\xi^0 \mathbf{c}(\xi, X)$ which vanishes by the previous Proposition. The result for forms follows from the product rule.

For densities, $\mathcal{L}_\xi^0 \mu = D_\xi^0 \mu$, and this means μ is basic as a section of $L_{\mathcal{H}}^1$. \square

The product rule means that other basic objects are characterized by vanishing Weyl-Lie derivative. For instance the submersion is conformal (i.e., the horizontal conformal structure is basic) iff $\mathcal{L}_\xi^0 \mathbf{c} = 0$.

Similarly a connection ∇ on E (associated to the frame bundle, or the conformal frame bundle if $\mathcal{L}_\xi^0 \mathbf{c} = 0$) is basic if it is horizontal and invariant with respect to \mathcal{L}_ξ^0 . In fact invariance suffices for the horizontal part of a connection ∇ to descend to B , but the pullback connection is then $\nabla - \langle \xi, \cdot \rangle \otimes \rho((D^0 \otimes \nabla)\xi)$.

The horizontal part of a Weyl derivative D on M is basic iff $0 = \mathcal{L}_\xi^0 D = F^D(\xi, \cdot) + \frac{1}{n}D^0(\text{div}^{D^0 \otimes D} \xi)$. If D is horizontal this reduces to $F^D(\xi, \cdot)$. In particular D^0 itself is basic iff $F^0(\xi, \cdot) = 0$. If $\mathcal{L}_\xi^0 \mathbf{c} = 0$, the only nonzero fundamental form is $\Omega = \Omega_{\mathcal{H}}$:

$$\langle \Omega(X, Y), \xi \rangle = 2\langle D_X^0 Y, \xi \rangle = -2\langle D_X^0 \xi, Y \rangle = -(d^0 \xi)(X, Y).$$

Note that $\mathcal{L}_\xi^0(d^0 \xi) = \iota_\xi(d^0)^2 \xi = F^0 - \xi \wedge (F^0(\xi, \cdot))$ and so Ω is basic iff $\xi \wedge F^0 = 0$.

Note also that the linearized Koszul formula with respect to the minimal Weyl derivative of a conformal submersion ξ reduces to:

$$(3.1) \quad (\mathcal{L}_\xi^0 D)_X = \llbracket \mathcal{L}_\xi^0 D^{L^1}, X \rrbracket + \frac{1}{2}\xi \triangle F^0(X) + \frac{1}{2}\langle \xi, X \rangle F^0.$$

Hence invariance on L^1 does not imply invariance on other natural bundles.

4. THE JONES-TOD CORRESPONDENCE

Suppose that ξ is a congruence on an oriented conformal 4-manifold M defining a conformal submersion π onto a manifold B . Let D^0 be the minimal Weyl derivative of ξ , define $\omega = -(*d^D \xi)(\xi, \cdot)$ (which can be computed using any Weyl derivative D) and let $D^{sd} = D^0 + \frac{1}{2}\omega$ and $D^B = D^0 + \omega$.

4.1. Remarks. The definition of ω uses the natural extension of the star operator to 2-forms of any weight. The star operator on 1-forms (of any weight) is a 3-form of the same weight defined by $\iota_X * \alpha = *(\langle X, \cdot \rangle \wedge \alpha)$ for any vector field X . In general the star operator on any manifold will be defined in terms of the orientation $*1$ so that a similar relation holds between the star operator, wedge product and interior multiplication, with no signs. As remarked in [7], this is more convenient in computations than the usual choice. Note that $*^2 = +1$ in four dimensions, whereas $*_B^2 = -1$ in three dimensions.

Since the star operator is an involution on 2-forms in four dimensions, $\Lambda^2 T^*M = \Lambda_+^2 T^*M \oplus \Lambda_-^2 T^*M$. The selfdual and antiselfdual parts of a 2-form are denoted $F = F_+ + F_-$. A skew endomorphism J of TM may be identified with a weightless 2-form $\Omega_J \in L^2 \Lambda^2 T^*M \cong L^{-2} \Lambda^2 TM$ via $\Omega_J(X, Y) = \langle JX, Y \rangle$ and J is said to be *selfdual* or *antiselfdual* if Ω_J is. If J is antiselfdual, then $\Omega_J = \eta \wedge J\eta - * \eta \wedge J\eta$ for any weightless unit 1-form η . It follows that, for a 1-form α (of any weight), $*\alpha = J\alpha \wedge \Omega_J$ for any antiselfdual endomorphism J with $J^2 = -id$. Note that if F is a selfdual 2-form, then $X \wedge F(Y) - Y \wedge F(X)$ is also selfdual.

4.2. Proposition. $\langle (D^0 \otimes D^{sd})\xi, \cdot \rangle$ is a selfdual 2-form of weight -1 .

Proof. In terms of an arbitrary Weyl derivative D ,

$$\begin{aligned} D^0 &= D - \frac{1}{4}(\operatorname{div}^{D^0 \otimes D} \xi) \xi + \frac{1}{2}(d^{D^0 \otimes D} \xi)(\xi, \cdot) \\ \omega &= -(*d^{D^0 \otimes D} \xi)(\xi, \cdot) \end{aligned}$$

and so $D^{sd} = D - \frac{1}{4}(\operatorname{div}^{D^0 \otimes D} \xi) \xi + \frac{1}{2}(d^{D^0 \otimes D} \xi)(\xi, \cdot) - \frac{1}{2}(*d^{D^0 \otimes D} \xi)(\xi, \cdot)$.

Substituting $D = D^{sd}$ into this formula gives the result. \square

4.3. Remark. This property clearly characterizes D^{sd} . One can characterize D^B in a similar way by the vanishing of the trace of $D^B \xi$ and the selfduality of $\operatorname{alt} D^B \xi$.

4.4. Proposition. The Weyl derivative D^B is basic iff D^0 has selfdual Faraday curvature.

Proof. Since $\omega(\xi) = 0$, D^B is basic iff $F^B(\xi, \cdot) = 0$, where F^B is the Faraday curvature of D^B . Now $D^B = D^0 + \omega$ and so

$$F^B(\xi, \cdot) = F^0(\xi, \cdot) + d\omega(\xi, \cdot) = F^0(\xi, \cdot) + \mathcal{L}_\xi^0 \omega.$$

Writing $\omega = -(*d^0 \xi)(\xi, \cdot) = -*(\xi \wedge d^0 \xi)$ yields

$$\mathcal{L}_\xi^0 \omega = -*(\xi \wedge \mathcal{L}_\xi^0 d^0 \xi) = -*(\xi \wedge F^0) = -(*F^0)(\xi, \cdot).$$

So $F^B(\xi, \cdot) = (F^0 - *F^0)(\xi, \cdot)$. Since $F^0 - *F^0$ is antiselfdual, this contraction with ξ vanishes iff $F^0 = *F^0$. \square

When D^0 has selfdual Faraday curvature, the conformal submersion is said to be *selfdual*. In this case D^B is a basic Weyl derivative on $L^1 \cong \pi^*L_B^1$ and the induced Weyl structure on B is sometimes called the *Jones-Tod Weyl structure*. It follows from the Koszul formula that the induced Weyl connection on TB pulls back to the conformal connection on $\mathcal{H} \cong \pi^*TB$ given by the horizontal part of the Weyl connection induced by D^B on TM . The same observation holds for $L^{-1}\mathcal{H} \cong \pi^*L_B^{-1}TB$.

Now observe that the map $\Omega_J \mapsto J\xi$ is an isomorphism from $L^2\Lambda_-^2 T^*M$ to $L^{-1}\mathcal{H}$ with inverse $\chi \mapsto \xi \wedge \chi - *(\xi \wedge \chi) \in L^{-2}\Lambda_-^2 TM \cong L^2\Lambda_-^2 T^*M$. If $J^2 = -id$ then $|J\xi| = 1$ and so this isomorphism is an isometry up to a constant multiple (conventionally, $|\Omega_J|^2 = 2$ when $J^2 = -id$). Also note that $\llbracket J_1, J_2 \rrbracket \xi = -2*(\xi \wedge J_1 \xi \wedge J_2 \xi)$.

The action of $\mathfrak{co}(TM)$ on antiselfdual endomorphisms is by commutator, and so only the antiselfdual part contributes, since selfdual and antiselfdual endomorphisms commute. Therefore Proposition 4.2 shows that $\mathcal{L}_\xi^0 J = D_\xi^{sd} J$, i.e., D^{sd} is horizontal on $L^2\Lambda_-^2 T^*M$. The linearized Koszul formula may be used to show that if ξ is selfdual, then D^{sd} is invariant on $L^2\Lambda_-^2 T^*M$. More precisely,

$$\mathcal{L}_\xi^0 D_X^{sd} = \frac{1}{2}F^0(\xi, X)id + \frac{1}{2}\xi \triangle F^0(X) - \frac{1}{2}X \triangle F^0(\xi) + \frac{1}{2}\langle \xi, X \rangle F^0,$$

which has selfdual skew part. Hence D^{sd} is basic on $L^2\Lambda_-^2 T^*M \cong \pi^*L_B^{-1}TB$. The following Proposition identifies it with D^B (which gives another proof of invariance).

4.5. Proposition. *If J is an antiselfdual endomorphism, then for any horizontal vector field X , $(D_X^B(J\xi))^{\mathcal{H}} = (D_X^{sd}J)\xi$.*

Proof. Write $\chi = J\xi$ so that $\Omega_J = \xi \wedge \chi - *(\xi \wedge \chi)$. Then

$$D_X^{sd}\Omega_J = D_X^{sd}\xi \wedge \chi + \xi \wedge D_X^{sd}\chi - *(D_X^{sd}\xi \wedge \chi + \xi \wedge D_X^{sd}\chi).$$

Now $D^{sd} = D^0 + \frac{1}{2}\omega = D^B - \frac{1}{2}\omega$ and so

$$D_X^{sd}\xi = \frac{1}{2}*(X \wedge \xi \wedge \omega),$$

$$D_X^{sd}\chi = D_X^B\chi + \frac{1}{2}\omega(\chi)X - \frac{1}{2}\langle \chi, X \rangle \omega.$$

Therefore $D_X^{sd}\xi \wedge \chi = -\frac{1}{2}\langle \chi, X \rangle *(\xi \wedge \omega) + \frac{1}{2}\omega(\chi) *(\xi \wedge X)$

and $\xi \wedge D_X^{sd}\chi = \xi \wedge D_X^B\chi + \frac{1}{2}\omega(\chi)\xi \wedge X - \frac{1}{2}\langle \chi, X \rangle \xi \wedge \omega$

which gives $\xi \wedge D_X^{sd}\chi - *(D_X^{sd}\xi \wedge \chi) = \xi \wedge D_X^B\chi$.

Taking the antiselfdual part and contracting with ξ completes the proof. \square

A generalized Jones-Tod correspondence follows readily from these observations, following an approach due to Gauduchon. Recall [13] that a Weyl connection is said to be *Einstein-Weyl* iff the symmetric traceless part of its Ricci tensor vanishes.

4.6. Theorem I. *Suppose (M, c) is an oriented conformal 4-manifold and ξ is a selfdual conformal submersion over a manifold B . Then $D^B = D^0 + \omega$ is Einstein-Weyl on B if and only if c is selfdual.*

Proof. Since ξ is selfdual, D^B descends to a Weyl connection on B . If π^*D^B denotes the pullback of D^B to $\pi^*L_B^{-1}TB \cong L^{-1}\mathcal{H}$ then Proposition 4.5 implies that $D^{sd}J = (\pi^*D^B)(J\xi)$ for any antiselfdual endomorphism J , and so $[[R_{X,Y}^{sd}, J]]\xi = (\pi^*R^B)_{X,Y}(J\xi)$, where R^B is the curvature of D^B on B and X, Y are arbitrary vector fields. Here I have used the fact that D^{sd} is horizontal, and so $D_\xi^{sd}J = \mathcal{L}_\xi^0 J = (\pi^*D^B)_\xi(J\xi)$ by definition of pullback; horizontality and the definition of pullback likewise imply that $[[R_{X,Y}^{sd}, J]]$ and $(\pi^*R^B)_{X,Y}(J\xi)$ vanish if X or Y is vertical. (Recall that $\mathcal{L}_\xi^0 D^{sd} = R_{\xi,X}^{sd} + (D^0 \otimes D^{sd})_X(D^0 \otimes D^{sd})\xi$.)

Let $R^{sd,-} : \Lambda^2 TM \rightarrow L^2 \Lambda^2 T^*M$ denote the antiselfdual part of R^{sd} and let $R^{B,0} : \Lambda^2 TB \rightarrow L_B^2 \Lambda^2 T^*B$ denote the skew part of R^B . Then, omitting pullbacks, $R^{sd,-}(J)\xi = -\frac{1}{2}*_B R^{B,0}(*_B J\xi)$, since $R^{sd,-}(\xi \wedge X) = 0$. The symmetric traceless part of $J \mapsto R^{sd,-}(J)$ is the antiselfdual Weyl tensor W^- and the symmetric traceless part of $\chi \mapsto *_B R^{B,0}(*_B \chi)$ is the symmetric traceless Ricci tensor of D^B , which proves the theorem. \square

An explicit formula for the relationship between r^{sd} and r^B will be useful later.

4.7. Proposition. *Suppose that ξ is a selfdual conformal submersion from a selfdual space M to an Einstein-Weyl space B . Then $F_-^{sd} = \frac{1}{4}(F^B - *F^B)$ and*

$$\text{sym } r^{sd} = \frac{1}{12} \text{scal}^B(\text{id} - 2\xi \otimes \xi) + \frac{1}{4}(*_B F^B \otimes \xi + \xi \otimes *_B F^B).$$

Proof. Let X, Y be basic vector fields. Since $\langle R^{sd,-}(\xi \wedge X), \xi \wedge Y \rangle = 0$, it follows that

$$r^{sd}(X, Y) + r^{sd}(\xi, \xi)\langle X, Y \rangle + *(\xi \wedge r^{sd}(\xi, \cdot) \wedge X \wedge Y) = 0.$$

On the other hand, since $\langle R^{sd}(*\xi \wedge X), \xi \wedge Y - *(\xi \wedge Y) \rangle = \langle R^{B,0}(*_B JX), *_B JY \rangle$, it follows that

$$\begin{aligned} r^{sd}(X, Y) - (\text{tr}_{\mathcal{H}} r^{sd})\langle X, Y \rangle - *(\xi \wedge r^{sd}(\cdot, \xi) \wedge X \wedge Y) \\ = r^B(X, Y) - (\text{tr } r^B)\langle X, Y \rangle = -\frac{1}{2}F^B(X, Y) - \frac{1}{6}\text{scal}^B\langle X, Y \rangle. \end{aligned}$$

The stated formulae follow easily from these. \square

Note that $F^{sd} = \frac{1}{2}(F^0 + F^B)$ and so the formula for F_-^{sd} follows immediately from the fact that $F_-^0 = 0$. On the other hand F^0 and F_+^{sd} are not basic in general.

The form of the Jones-Tod construction stated in Theorem I gives a procedure for constructing Einstein-Weyl spaces from selfdual spaces. For the inverse construction, the following reformulation is useful.

4.8. Proposition. *Suppose that (M, \mathfrak{c}) is an oriented conformal 4-manifold, that ξ is a conformal submersion over an Einstein-Weyl manifold B , and that $D^0 = \pi^*D^B - \omega$, where D^B is the Weyl derivative on L_B^1 and $\omega = -(*d^D\xi)(\xi, \cdot)$ (computed using any Weyl derivative D). Then (M, \mathfrak{c}) is selfdual and ξ is selfdual.*

This follows immediately from Theorem I and Proposition 4.4: $\pi^*D^B - \omega$ has selfdual Faraday curvature since π^*D^B is basic. Therefore, an inverse Jones-Tod construction will be obtained if the equation $D^0 = \pi^*D^B - \omega$ can be interpreted

as an equation on B . In order to do this, recall from Proposition 3.2, that a conformal structure on a fibre bundle $\pi: M \rightarrow B$ over a conformal manifold B is determined by a connection 1-form $\eta: TM \rightarrow \mathcal{V}$ and a relative length scale $w: \pi^*L_B^1 \rightarrow \mathcal{V}$, where I have assumed the fibres are oriented and one dimensional so that $L_{\mathcal{V}}^1 = \mathcal{V}$. Choosing a fibre coordinate t identifies M locally with $B \times \mathbb{R}$, providing a trivialization of \mathcal{V} and a flat connection 1-form dt . In these terms $\eta = dt + A$ for $A \in C^\infty(B \times \mathbb{R}, \pi^*T^*B)$ and $w \in C^\infty(B \times \mathbb{R}, \pi^*L_B^{-1})$.

The inverse Jones-Tod construction can now be formulated as a nonlinear evolution equation on B .

4.9. Proposition. *Let (M, c) be a selfdual conformal 4-manifold with a selfdual conformal submersion ξ over an Einstein-Weyl space (B, c_B, D^B) . Then $M \rightarrow B$ is locally conformal to $\pi: B \times \mathbb{R} \rightarrow B$ with conformal structure*

$$c = \pi^*c_B + w^{-2}(dt + A)^2$$

where
$$*_B(D^B w + \dot{A}w - A\dot{w}) = dA + \dot{A} \wedge A,$$

for $w \in C^\infty(B \times \mathbb{R}, L_B^{-1})$ and $A \in C^\infty(B \times \mathbb{R}, T^*B)$. Here t is the fibre coordinate on $B \times \mathbb{R}$ and w and A are viewed as a time-dependent density and 1-form on B , so that a dot denotes differentiation with respect to t , while $D^B w$ and dA are the derivatives on B .

Conversely for any solution of these equations on an Einstein-Weyl space B , the conformal structure given by the above formula is selfdual, and π defines a selfdual conformal submersion over B .

Proof. A conformal submersion certainly has the form given. The aim of the proof is to show that the given equations on B are equivalent to the fact that $\pi^*D^B = D^0 + \omega$ on $B \times \mathbb{R}$. To do this, I will work in the (arbitrarily chosen) gauge $g = w^2 c$ and rewrite the equation $\pi^*D^B = D^0 + \omega = D^g - \frac{1}{3}(\text{div}^g \xi)\xi + (d^g \xi)(\xi, \cdot) - (*d^g \xi)(\xi, \cdot)$ using the fact that $D^g w = 0$.

In the chosen gauge, $w\xi = dt + A$ and so

$$wd^g \xi = d(w\xi) = dA + dt \wedge \dot{A} = w\xi \wedge \dot{A} + dA + \dot{A} \wedge A.$$

It follows that $(d^g \xi)(\xi, \cdot) = \dot{A}$ and

$$(*d^g \xi)(\xi, \cdot) = *(\xi \wedge d^g \xi) = -*_B(dA + \dot{A} \wedge A).$$

Writing div^g in terms of $*d^g*$ readily yields $\frac{1}{3}\text{div}^g \xi = \dot{w}$. Therefore:

$$\begin{aligned} 0 &= D^g w = (\pi^*D^B)w - \dot{w}w\xi + \dot{A}w + *_B(dA + \dot{A} \wedge A) \\ &= D^B w + \dot{w}dt - \dot{w}(dt + A) + \dot{A}w + *_B(dA + \dot{A} \wedge A) \\ &= D^B w - A\dot{w} + \dot{A}w + *_B(dA + \dot{A} \wedge A). \end{aligned}$$

This completes the proof and also shows that $D^0 = D^g - \dot{w}\xi + \dot{A}$. \square

4.10. Remarks. These equations may be viewed as Einstein-Weyl Bogomolny equations for a diffeomorphism group: if the fibres are diffeomorphic to an oriented 1-manifold \mathbb{T} (S^1 or \mathbb{R} , assuming the fibres are connected), then $M = P \times_{\text{Diff}(\mathbb{T})} \mathbb{T}$

where $\text{Diff}(\mathbb{T})$ is the group of orientation preserving diffeomorphisms of \mathbb{T} and P is the principal $\text{Diff}(\mathbb{T})$ -bundle whose fibre at $x \in B$ consists of the orientation preserving diffeomorphisms $\mathbb{T} \rightarrow M_x$. Note that P and $\text{Diff}(\mathbb{T})$ only enter into this formulation infinitesimally, so one can assume that M is an open subset of $P \times_{\text{Diff}(\mathbb{T})} \mathbb{T}$. Choosing a gauge, i.e., a (local) section of P , identifies M (locally) with $B \times \mathbb{T}$ and the connection 1-form and relative length scale may be viewed as sections of $L_B^{-1} \otimes \text{Vect}(\mathbb{T})$ and $T^*B \otimes \text{Vect}(\mathbb{T})$, where $\text{Vect}(\mathbb{T})$ is the Lie algebra of vector fields on \mathbb{T} . If t is a coordinate on \mathbb{T} , then writing $w = w(t) d/dt$ and $A = A(t) d/dt$ shows that the equations of the above proposition are:

$$*_B(D^B w + [A, w]) = F^A := dA + \frac{1}{2}[A \wedge A]$$

where $[\cdot, \cdot]$ denotes the Lie bracket in $\text{Vect}(\mathbb{T})$.

The classical Jones-Tod construction arises by reduction to a one dimensional translational subgroup S^1 or \mathbb{R} —this point of view will be further justified later by studying the other finite dimensional subgroups of $\text{Diff}(\mathbb{T})$.

The gauge freedom is of course the choice of t coordinate for these monopole equations. If $\tilde{t} = f(t)$ for a function f on $B \times \mathbb{T}$ with $\dot{f} \neq 0$ then $w(t) = \tilde{w}(\tilde{t})/\dot{f}$ and $A(t) = (\tilde{A}(\tilde{t}) + df)/\dot{f}$. In the classical Jones-Tod correspondence there is a preferred gauge in which to work: the constant length gauge of the conformal vector field K ($D^0 = D^{|K|}$ is exact). Choosing a section $t = 0$ of M over B makes it into a line bundle and M may be recovered from a linear differential equation on B :

$$c = \pi^* c_B + w^{-2}(dt + A)^2$$

where

$$*_B D^B w = dA \quad \text{for} \quad w \in C^\infty(B, L_B^{-1}).$$

This equation for (w, A) is often called the (abelian) monopole equation.

Now suppose that J is an antiselfdual almost complex structure on M which is invariant with respect to ξ , i.e., $\mathcal{L}_\xi^0 J = 0$. Let $\Omega_J(X, Y) = \langle JX, Y \rangle$ be the conformal Kähler form of J and D the unique Weyl derivative such that $d^D \Omega_J = 0$. Then it is well known that J is integrable if and only if $DJ = 0$; D is then called the *Kähler-Weyl* connection of J . If $DJ = 0$ then $(\mathcal{L}_\xi^0 J)X = J(D^0 \otimes D)_X \xi - (D^0 \otimes D)_{JX} \xi$, and so J is invariant iff ξ is *holomorphic* in the sense that $(D^0 \otimes D)\xi$ is complex linear. The following generalizes a theorem of [7] to conformal submersions.

4.11. Theorem II. *Let M be an oriented conformal 4-manifold with a selfdual conformal submersion ξ over a manifold B and suppose that J is an invariant antiselfdual almost complex structure on M . Let $D = D^{sd} - \kappa\xi - \tau\chi$ for basic sections τ and κ of L^{-1} , where $\chi = J\xi$. Then $DJ = 0$ iff $D^B \chi = \tau(\text{id} - \chi \otimes \chi) + \kappa *_B \chi$ on B . Hence J is integrable iff χ is a shear-free geodesic congruence, in the sense that $D^B \chi$ has the above form.*

Proof. Since J is invariant, χ is invariant, hence basic, since it is horizontal. Note that $D_\xi J = D_\xi^{sd} J - \tau[\xi \triangle \chi, J] = D_\xi^{sd} J = \mathcal{L}_\xi^0 J = 0$, so it remains to compute $D_X J$

for horizontal vector fields X . Since $\Omega_J = \xi \wedge \chi - *(\xi \wedge \chi)$ it follows that

$$D_X \Omega_J = D_X \xi \wedge \chi + \xi \wedge D_X \chi - *(D_X \xi \wedge \chi + \xi \wedge D_X \chi),$$

where

$$D_X \xi \wedge \chi = D_X^{sd} \xi \wedge \chi - \kappa X \wedge \chi$$

and

$$\xi \wedge D_X \chi = \xi \wedge D_X^{sd} \chi - \tau \xi \wedge (X - \langle \chi, X \rangle \chi).$$

Therefore

$$\xi \wedge D_X \chi - *(D_X \xi \wedge \chi) = \xi \wedge D_X^B \chi - \tau \xi \wedge (X - \langle \chi, X \rangle \chi) + \kappa *(X \wedge \chi).$$

Since the right hand side is a vertical 2-form, it follows that $D_X J = 0$ iff

$$D_X^B \chi - \langle D_X^B \chi, \xi \rangle = \tau (X - \langle \chi, X \rangle \chi) + \kappa \iota_X *B\chi.$$

To prove the final statement, suppose that J is invariant and integrable with Kähler-Weyl connection D . Then $(D^0 \otimes D)\xi = -\kappa \text{id} + \frac{1}{2}\tau J + \frac{1}{2}(d^{D^0 \otimes D}\xi)^+$, where $(d^{D^0 \otimes D}\xi)^+$ is a selfdual 2-form and κ, τ are sections of L^{-1} . It follows that

$$(d^{D^0 \otimes D}\xi)(\xi, \cdot) = \tau \chi + (d^{D^0 \otimes D}\xi)^+(\xi, \cdot)$$

$$(*d^{D^0 \otimes D}\xi)(\xi, \cdot) = -\tau \chi + (d^{D^0 \otimes D}\xi)^+(\xi, \cdot).$$

Therefore $D^{sd} = D + \kappa \xi + \tau \chi$. It remains to check that κ and τ are basic. Since $DJ = 0$ and $\mathcal{L}_\xi^0 J = 0$ it follows that $[\mathcal{L}_\xi^0 D_X, J] = 0$. By the linearized Koszul formula, this implies $\mathcal{L}_\xi^0 D = \frac{1}{2}F^0(\xi)$ on L^1 and therefore

$$\begin{aligned} R_{\xi, X}^D + (D^0 \otimes D)_X \left(-\kappa \text{id} + \frac{1}{2}\tau J + \frac{1}{2}(d^{D^0 \otimes D}\xi)^+ \right) \\ = \mathcal{L}_\xi^0 D_X = \frac{1}{2}F^0(\xi, X)\text{id} + \frac{1}{2}F^0(\xi) \triangle X - \frac{1}{2}F^0(X) \triangle \xi + \frac{1}{2}\langle \xi, X \rangle F^0. \end{aligned}$$

The identity and J components of this formula give $D^0 \kappa + \frac{1}{2}F^0(\xi) = F^D(\xi)$ and $D^0 \tau = \rho^D(\xi)$ where $\rho^D(X, Y)$ is the Ricci form of D , defined to be the contraction of $R_{X, Y}^D$ with J . In particular $D_\xi^0 \kappa = 0 = D_\xi^0 \tau$. \square

4.12. Remark. τ and κ are called the *divergence* and *twist* of the congruence χ .

Assume now that W is selfdual. Then so are F^D and ρ^D (see e.g. [7]), and hence they are uniquely determined by their contractions with ξ . It follows that (M, J) is locally hypercomplex iff $D^0 \tau = 0$ iff $\tau = 0$ or D^0 is exact and τ is constant in this gauge. This implies that a hyperCR structure on the Einstein-Weyl quotient B induces a hypercomplex structure on M , generalizing a result of Gauduchon and Tod [11] to conformal submersions. On the other hand, (M, J) is locally scalar-flat Kähler iff $D^0 \kappa + \frac{1}{2}F^0(\xi) = 0$, so the presence of F^0 obstructs a naive generalization of LeBrun's work [17] to this context. This will be remedied in the next section.

I next generalize a result of Mason and Tod, which was used by Tod [22] to give a general description of selfdual Einstein metrics with a Killing vector field.

4.13. Theorem III. *Let $(M, \mathfrak{c}, D^{ew})$ be a selfdual Einstein-Weyl 4-manifold and let ξ a selfdual conformal submersion with minimal Weyl derivative D^0 . Then M admits a canonical compatible Kähler-Weyl structure on the open set where the antiselfdual part of $(D^0 \otimes D^{ew})\xi$ is nonzero.*

More precisely, if this antiselfdual part is τJ where $J^2 = -\text{id}$ then J is integrable, with Kähler-Weyl connection $D = D^{ew} - \tau^{-1}D^0\tau = D^{ew} + D^0 - D^\tau$, where D^τ is defined by $D^\tau\tau = 0$.

Proof. It suffices to prove that $DJ = 0$. Observe first that $(D_X^0\tau)J + \tau D_X^{ew}J$ is the antiselfdual part of $(D^0 \otimes D^{ew})_X(D^0 \otimes D^{ew})\xi = \mathcal{L}_\xi^0 D_X^{ew} - R_{\xi,X}^{ew}$, where R^{ew} is the curvature of D^{ew} . Expanding the curvature and using the linearized Koszul formula to compute the Weyl-Lie derivative gives

$$\begin{aligned} & (D_X^0\tau)J + \tau D_X^{ew}J \\ &= \left[\frac{1}{2}F^{ew}(\xi) \triangle X + \frac{1}{2}F^{ew}(X) \triangle \xi + \frac{1}{12} \text{scal}^{ew} \xi \triangle X - D^0\kappa \triangle X + \frac{1}{2}\xi \triangle F^0(X) \right]^- \\ &= \left[(F^{ew}(\xi) + \frac{1}{12} \text{scal}^{ew} \xi - D^0\kappa - \frac{1}{2}F^0(\xi)) \triangle X \right]^- , \end{aligned}$$

where $[...]^-$ denotes the antiselfdual part, κ is minus the identity component of $(D^0 \otimes D^{ew})\xi$, scal^{ew} is the scalar curvature of D^{ew} , F^{ew} is the Faraday curvature of D^{ew} , and I have used the fact that F^0 and F^{ew} are selfdual. The precise form of this expression is now not important: it suffices to observe that it is of the form $[J\alpha \triangle X]^-$ for some 1-form α . Since $*(J\alpha \triangle X) = -(\alpha \triangle JX + \alpha(X)J)$, it follows that $J\alpha \triangle X - *(J\alpha \triangle X) = \llbracket \alpha \triangle X, J \rrbracket + \alpha(X)J$ and the commutator term is orthogonal to J . Since $D^{ew}J$ is also orthogonal to J , $D_X^0\tau = \alpha(X)$ and $\tau D_X^{ew}J = \llbracket \alpha \triangle X, J \rrbracket = \llbracket D^0\tau \triangle X, J \rrbracket$, i.e., J is parallel with respect to $D^{ew} - \tau^{-1}D^0\tau$. \square

The theorems of this section reduce to known results when D^0 is exact (i.e., $\xi = K/|K|$ for some conformal vector field K), but they have one disadvantage over the results they generalize. Namely, the abelian monopole equation on B arising in the classical Jones-Tod correspondence becomes a nonlinear evolution equation, which is much harder to solve. This difficulty has already been encountered in another special case of the above theorems: the case that M is the selfdual Einstein metric locally “filling in” B via Hitchin’s version of LeBrun’s \mathcal{H} -space construction [13, 16]. This beautiful construction of a selfdual Einstein metric from a real analytic conformal 3-manifold (which is taken to be Einstein-Weyl in Hitchin’s construction) is defined twistorially, making it difficult to carry out in practice.

Furthermore conformal submersions themselves are hard to find, because the equation for conformal submersions, unlike the conformal Killing equation, is nonlinear. Hence, for the theorems of this section to be interesting, it is essential to find new situations in which the inverse construction can be carried out and examples can be found. This will be done in the next two sections. In the final section, a direct version of the Hitchin-LeBrun construction will be obtained.

5. AFFINE CONFORMAL SUBMERSIONS

An affine structure on a submersion $\pi: M \rightarrow B$ is a flat torsion-free connection on each fibre. This identifies M , at least locally, with an affine bundle modelled on the vector bundle whose fibre at each point of B is the space of parallel vector fields on the corresponding fibre of M . For conformal submersions with oriented one dimensional fibres, the vertical bundle \mathcal{V} of M is isomorphic to L^1 and so the vertical part D_ξ of any Weyl derivative D defines an affine structure on M .

There are many choices of affine structure on M , but such a choice is only helpful if the conformal structure on M is affine; that is, in terms of Proposition 3.2, the nonlinear connection on M is an affine connection, and the relative length scale is affine along the fibres. If such a “good” affine structure can be found, I will say that π is an *affine conformal submersion*.

More precisely, a nonlinear connection \mathcal{H} induces a linearized connection on the infinite dimensional vector space of vertical vector fields defined by $\mathcal{D}_X U = [\tilde{X}, U]$, where U is a vertical vector field, X is a vector field on B and \tilde{X} is its horizontal lift, so that $[\tilde{X}, U]$ is vertical. \mathcal{H} is affine iff the parallel vertical vector fields on each fibre are preserved by \mathcal{D} ; this then induces the linearized connection on the model vector bundle. Affine connections form an affine space modelled on 1-forms on B with values in the affine vector fields on \mathcal{V} . Similarly, the relative length scale $w: \pi^* L_B^1 \rightarrow \mathcal{V}$ is affine iff it maps basic densities to affine vector fields, in which case it may be viewed as a (-1) -density on B with values in the affine vector fields.

The above approach and the next proposition arose from discussions with Paul Gauduchon in a joint effort to understand affine conformal submersions.

5.1. Proposition. *Let M be a conformal manifold and let ξ be a conformal submersion over B with minimal Weyl derivative D^0 . Define an affine structure D_ξ on $\pi: M \rightarrow B$ by the Weyl derivative $D = D^0 + \lambda\xi$ where λ is a section of L^{-1} .*

- (i) *The connection \mathcal{H} on $M \rightarrow B$ is affine with respect to D_ξ iff $F^D(\xi) = 0$.*
- (ii) *The relative length scale is affine with respect to D_ξ iff λ is basic.*

Hence the conformal submersion is affine if $D^0\lambda = F^0(\xi)$.

Proof. The D_ξ -parallel vertical vector fields are defined by identifying \mathcal{V} with L^1 using ξ . Hence the linearized connection may be defined on $\mu \in C^\infty(M, L^1)$ by

$$\begin{aligned} (\mathcal{D}_X \mu)\xi &= [\tilde{X}, \mu\xi] = D_{\tilde{X}}^0(\mu\xi) - \mu D_{\tilde{X}}^0 \tilde{X} = (D_{\tilde{X}}^0 \mu)\xi - \mu(D_{\tilde{X}}^0 \xi - D_\xi^0 \tilde{X}) \\ &= (D_{\tilde{X}}^0 \mu)\xi + \mathcal{L}_\xi^0 \tilde{X} = (D_{\tilde{X}} \mu)\xi \end{aligned}$$

since \tilde{X} is invariant and $D - D^0$ is vertical. Hence \mathcal{H} is affine iff $D_\xi(D_{\tilde{X}} \mu) = 0$ for all μ with $D_\xi \mu = 0$. Since $[U, \tilde{X}]$ is vertical for U vertical, $D_\xi(D_{\tilde{X}} \mu) = F^D(\xi, \tilde{X})\mu$ and so \mathcal{H} is affine iff $F^D(\xi) = 0$.

The relative length scale is section w of $\pi^* L_B^{-1} \otimes \mathcal{V}$. This is affine iff its vertical derivative with respect to the affine structure, as a section of $\pi^* L_B^{-1} \otimes \mathcal{V}^* \otimes \mathcal{V} \cong \pi^* L_B^{-1}$, is basic. Identifying $\pi^* L_B^1$ and \mathcal{V} with L^1 identifies w with the identity map in $L^{-1} \otimes L^1$ but its vertical derivative must be computed with respect to the covariant derivative $D^0 \otimes D$ and so w is affine iff $0 = D_\xi^0(D^0 \otimes D)_\xi id = D_\xi^0 \lambda$.

Now observe that $F^D(\xi) = F^0(\xi) + d(\lambda\xi)(\xi) = F^0(\xi) + (D_\xi^0 \lambda)\xi - D^0 \lambda$. \square

It follows from this that there is an obstruction to the existence of a good affine structure for a conformal submersion: since $-F^0 \lambda = d^0(F^0(\xi)) = \mathcal{L}_\xi^0 F^0$, the Weyl-Lie derivative of F^0 must be a multiple of F^0 . If a good affine structure exists, it is essentially unique: any two differ by a section μ of L^{-1} with $D^0 \mu = 0$ which implies that the affine structures are equal or D^0 is exact and μ is constant.

I now return to four dimensions and the Jones-Tod construction.

5.2. Theorem IV. *Let (M, \mathbf{c}) be a selfdual conformal 4-manifold with a selfdual affine conformal submersion $\pi: M \rightarrow B$ over an Einstein-Weyl space (B, \mathbf{c}_B, D^B) . Then with respect to an arbitrary affine coordinate t on $M \rightarrow B$, the conformal structure on M is*

$$\mathbf{c} = \pi^* \mathbf{c}_B + (tw_1 + w_0)^{-2} (dt + tA_1 + A_0)^2$$

where

$$*_B D^B w_1 = dA_1$$

and

$$*_B (D^B w_0 + A_1 w_0 - A_0 w_1) = dA_0 + A_1 \wedge A_0$$

for some $w_0, w_1 \in C^\infty(B, L_B^{-1})$ and $A_0, A_1 \in C^\infty(B, T^*B)$. Conversely for any solution of these affine monopole equations on an Einstein-Weyl space B , the conformal structure given by the above formula is selfdual, and the above decomposition defines a selfdual affine conformal submersion over B .

Proof. An affine conformal submersion certainly has the form given. It remains to apply this Ansatz to the equations of Proposition 4.9, by writing $w = tw_1 + w_0$ and $A = tA_1 + A_0$. Now

$$\begin{aligned} D^B w + \dot{A}w - A\dot{w} &= tD^B w_1 + D^B w_0 + w_0 A_1 - w_1 A_0 \\ dA + \dot{A} \wedge A &= tdA_1 + dA_0 + A_1 \wedge A_0 \end{aligned}$$

and the linear and constant terms (in t) of these equations prove the result. \square

5.3. Remarks. Note that $D^0 = D^g - w_1 \xi + A_1$ and so $F^0(\xi) = -d(w_1 \xi)(\xi) = D^0 w_1 - (D_\xi^0 w_1) \xi = D^0 w_1$ since w_1 is basic. Hence $\lambda = w_1$ is the solution of $D^0 \lambda = F^0(\xi)$: the affine structure is $D_\xi^g = D_\xi^0 + w_1$.

The freedom in the choice of affine coordinate t gives a gauge freedom for the affine monopole equations. If $\tilde{t} = at + b$ for basic functions a, b , write $w_1 = \tilde{w}_1$, $w_0 = a^{-1}(\tilde{w}_0 + b\tilde{w}_1)$, $A_1 = \tilde{A}_1 + a^{-1}da$ and $A_0 = a^{-1}(\tilde{A}_0 + b\tilde{A}_1 + db)$. One immediately verifies, by substituting into the affine monopole equations, that (\tilde{w}, \tilde{A}) is a solution if (w, A) is. Note that $t\mu_g = tw^{-1}$ is a well defined section of L^1 up to translation by a basic section of L^1 : it may be fixed by choosing a section of $M \rightarrow B$. The induced exact Weyl derivative is $D^{t\mu_g} = D^g - t^{-1}dt = D^0 - t^{-1}w_0\xi + t^{-1}A_0$.

The equations show that the affine Jones-Tod correspondence reduces to the classical case in two ways. Firstly the linear part of the affine monopole equation is an abelian monopole equation: if (w_0, A_0) is zero, the affine bundle M is isomorphic to the model vector bundle. Secondly if the linear part (w_1, A_1) vanishes, the translational part of the of the affine monopole equation is an abelian monopole equation: the model vector bundle is trivial, and so M is a principal \mathbb{R} -bundle. On the other hand if the solution (w_1, A_1) is nontrivial, then it gives a linearization of M , i.e., a selfdual space with an affine conformal submersion is affinely modelled on a selfdual space with a conformal vector field.

This theorem gives a new method for constructing selfdual spaces from linear equations, since the second affine monopole equation is linear once a solution of the

first equation is chosen. In particular, this method gives all scalar-flat Kähler metrics with a holomorphic selfdual conformal submersion including all hyperKähler metrics admitting such a submersion.

5.4. Theorem V. *Let (M, g) be a four dimensional scalar-flat Kähler metric (with antiselfdual complex structure, so that (M, c) is selfdual) admitting a holomorphic selfdual conformal submersion. Then the conformal submersion is affine over an Einstein-Weyl space with a shear-free geodesic congruence χ , where the linear part of the affine monopole is given by $w_1 = -2\kappa$ and κ is the twist of the congruence χ . All such metrics are locally of the form*

$$g = (\rho - 2\mu_t^{-1}\kappa)c_B + \frac{(D^B(\mu_t^{-1}) + 2\tau\chi\mu_t^{-1} + \Phi)^2}{\rho - 2\mu_t^{-1}\kappa},$$

where $\rho \in C^\infty(B, L_B^{-2})$, $\Phi \in C^\infty(B, L_B^{-1}T^*B)$, μ_t^{-1} is a section of L^{-1} increasing along the fibres, and τ is the twist of the congruence χ . Conversely, for any Einstein-Weyl space (c_B, D^B) and shear-free geodesic congruence χ with twist κ and divergence τ , this metric is scalar-flat Kähler iff (ρ, Φ) satisfy the linear differential equation

$$*_B(D^B\rho + 2\tau\chi\rho + 2\kappa\Phi) = d^B\Phi + 2\tau\chi \wedge \Phi.$$

Furthermore the metric is hyperKähler iff $D^0\tau = 0$, i.e., iff $\tau = 0$ or D^0 is exact and τ is constant in this gauge.

Proof. By Theorem II, $D^g = D^{sd} - \kappa\xi - \tau\chi$ where $D^B\chi = \tau(id - \chi \otimes \chi) + \kappa *_B\chi$, and furthermore, $D^0\kappa + \frac{1}{2}F^0(\xi) = 0$, since $F^{D^g} = 0$. Hence $D_\xi^0 - 2\kappa$ defines an affine structure on M making the submersion affine and $w_1 = -2\kappa$.

The analysis of shear-free geodesic congruences in [7] shows that $*_BD^B\kappa = \frac{1}{2}F^B - d(\tau\chi)$, and hence, choosing any gauge μ_B , one can take $A_1 = -\omega_B + 2\tau\chi$. The translational part of the monopole equation is therefore:

$$*_B(D^Bw_0 - \omega_Bw_0 + 2\tau\chi w_0 + 2\kappa A_0) = dA_0 + (-\omega_B + 2\tau\chi) \wedge A_0.$$

Now the Levi-Civita connection of the affine gauge is $D^0 + w_1\xi - A_1 = D^0 - 2\kappa\xi - 2\tau\chi + \omega_B$, whereas the Levi-Civita connection of the μ_B -gauge is $D^B - \omega_B$. The barycentre of these is $D^{sd} - \kappa\xi - \tau\chi$, which is the Levi-Civita connection of the scalar-flat Kähler metric. This identifies g within the conformal class, and putting $\rho = \mu_B^{-1}w_0$, $\Phi = \mu_B^{-1}A_0$ and $\mu_t^{-1} = t\mu_B^{-1}$ completes the proof. \square

When $\kappa = 0$ this is LeBrun's construction of scalar-flat Kähler metrics with Killing fields [17]. On the other hand, when $(w_0, A_0) = 0$, this theorem reduces to the construction of scalar-flat Kähler metrics with homothetic vector fields [7], including, as a special case, the hyperKähler metrics of [11].

I end this section with some nontrivial examples. In [8], the following Einstein-Weyl structures were found from solutions of the $SU(\infty)$ Toda field equation.

$$g_B = (z + h)(z + \bar{h})g_{S^2} + dz^2, \quad \omega_B = -\frac{2z + h + \bar{h}}{(z + h)(z + \bar{h})}dz,$$

where h is a holomorphic function on an open subset of S^2 and $D^B = D^g + \omega$. Note that the weightless unit vector field dual to dz generates a shear-free geodesic congruence with vanishing twist ($\tau \neq 0, \kappa = 0$). These spaces also admit shear-free geodesic congruences with vanishing divergence ($\tau = 0, \kappa \neq 0$), i.e., they are hyperCR, and they are called the *hyperCR-Toda spaces*.

Applying the classical Jones-Tod construction to these spaces gives conformal structures of the form $c = \pi^*c_B + w^{-2}(\beta + v dz)^2$, where $\beta = dt + \theta$ for a 1-form θ on B orthogonal to dz , and $*_B D^B w = dA$ with $A = \theta + v dz$.

These conformal structures admit a compatible scalar-flat Kähler metric and also a compatible hypercomplex structure. The conformal vector field $\partial/\partial t$ is a Killing field of the scalar-flat Kähler metric and triholomorphic with respect to the hypercomplex structure.

For certain solutions of the abelian monopole equation, $\partial/\partial z$ defines a conformal submersion. To see this, write $c = \varepsilon_0^2 + \varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2$ where ε_0 and ε_3 are the weightless unit 1-forms corresponding to $w dz$ and $\beta + v dz$. The weightless unit 1-form dual to $\partial/\partial z$ is $\xi = (w\varepsilon_0 + v\varepsilon_3)/\sqrt{w^2 + v^2}$ and so $\varepsilon_0^2 + \varepsilon_3^2 - \xi^2 = (v\varepsilon_0 - w\varepsilon_3)^2/(w^2 + v^2) = w^2\beta^2/(w^2 + v^2)$. Hence if β and $(w^2 + v^2)|z + h|^2$ are independent of z , then $\partial/\partial z$ will define a conformal submersion with quotient $(w^2 + v^2)|z + h|^2 g_{S^2} + \beta^2$.

Now Ian Strachan has pointed out [8] that for any holomorphic function f ,

$$w = \frac{1}{2} \left(\frac{f}{z+h} + \frac{\bar{f}}{z+\bar{h}} \right), \quad v = \frac{1}{2i} \left(\frac{f}{z+h} - \frac{\bar{f}}{z+\bar{h}} \right), \quad d\beta = \frac{1}{2}(f + \bar{f}) \text{vol}_{S^2}$$

defines a solution of the monopole equation. Clearly β and $(w^2 + v^2)|z + h|^2$ are independent of z and so $\partial/\partial z$ defines a conformal submersion. Explicitly, c has a compatible metric

$$g = \left(\frac{z+h}{2f} + \frac{z+\bar{h}}{2\bar{f}} \right)^2 (|f|^2 g_{S^2} + \beta^2) + \left[dz + i \left(\frac{z+h}{2f} - \frac{z+\bar{h}}{2\bar{f}} \right) \beta \right]^2$$

and so this submersion is obviously affine, with affine coordinate z . The quotient conformal 3-manifold \tilde{B} admits an Einstein-Weyl structure:

$$g_{\tilde{B}} = |f|^2 g_{S^2} + \beta^2, \quad \omega_{\tilde{B}} = \frac{i}{2} \left(\frac{1}{f} - \frac{1}{\bar{f}} \right) \beta.$$

These are the Einstein-Weyl spaces with *geodesic symmetry* described in [7]. One easily checks that g is given by a solution of the affine monopole equations with $w_1 = -2\kappa_s$ where κ_s is the twist of the geodesic symmetry on \tilde{B} , i.e., $d\beta = 2\kappa_s *_B \beta$. Hence these scalar-flat Kähler metrics with compatible hypercomplex structures could have been constructed directly as selfdual affine conformal submersions over the Einstein-Weyl spaces with geodesic symmetry. When $f = ah + b$ for $a, b \in \mathbb{R}$, these metrics are conformally Einstein [8] and will feature again in the final section.

6. PROJECTIVE CONFORMAL SUBMERSIONS

A natural generalization of an affine conformal submersion is a projective conformal submersion. A projective structure on a 1-manifold is a second order linear

differential operator from $L^{1/2}$ to $L^{-3/2}$ which has no first order term with respect to any Weyl derivative, and the same definition may be applied fibrewise to a congruence ξ . Hence any Weyl derivative D induces a projective structure $\mu \mapsto D_\xi(D_\xi\mu)$. Note that $(D + \gamma)_\xi((D + \gamma)_\xi\mu) = (D_\xi - \frac{1}{2}\gamma(\xi))(D_\xi\mu + \frac{1}{2}\gamma(\xi)\mu) = D_\xi(D_\xi\mu) + \frac{1}{2}D_\xi(\gamma(\xi))\mu - \frac{1}{4}(\gamma(\xi))^2\mu$, verifying that the condition of vanishing first order term is independent of the Weyl derivative.

A conformal submersion π will be called *projective* iff there is a projective structure on π such that the connection \mathcal{H} is projective and the relative length scale w takes values in the projective vector fields: recall that these are characterized as being quadratic in any projective coordinate.

6.1. Remark. A curve in a conformal manifold, with weightless unit tangent ξ has a canonical projective structure given by $(D_\xi)^2 + \frac{1}{2}r^D(\xi, \xi) + \frac{1}{4}|D_\xi\xi|^2$, and is called a conformal geodesic if $D_\xi(D_\xi\xi) + |D_\xi\xi|^2\xi - r^D(\xi) + r^D(\xi, \xi)\xi = 0$; these expressions are independent of the Weyl derivative D . However, if ξ is a projective conformal submersion, there is no reason for the projective structure to equal the canonical one, nor will the fibres be conformal geodesics in general.

6.2. Proposition. *Let M be a conformal manifold and let ξ be a conformal submersion over B with minimal Weyl derivative D^0 . Then the conformal submersion is projective with respect to the projective structure $(D_\xi^0)^2 + \frac{1}{2}\rho$, for a section ρ of L^{-2} , iff $D^0\rho = \mathcal{L}_\xi^0 F^0(\xi)$.*

Proof. Write the projective structure as $(D_\xi)^2$ where $D_\xi = D_\xi^0 + \lambda$ is a compatible affine structure, so that $\rho = D_\xi^0\lambda - \frac{1}{2}\lambda^2$. Then the connection \mathcal{H} is projective iff it maps D_ξ -parallel vertical vector fields to D_ξ -affine vertical vector fields, i.e., iff $D_\xi\mu = 0 \implies \partial_\xi(D_\xi(D_X\mu)) = 0$ for basic vector fields X . This condition reduces easily to $(\mathcal{L}_\xi^0 - \lambda)(F^0(\xi, X) - D_X^0\lambda) = 0$.

The relative length scale is projective iff $0 = (D^0 \otimes D)_\xi(D_\xi^0\lambda) = (\mathcal{L}_\xi^0 - \lambda)(D_\xi^0\lambda)$. Hence the conformal submersion is projective iff

$$\begin{aligned} 0 &= (\mathcal{L}_\xi^0 - \lambda)(F^0(\xi) - D^0\lambda) \\ &= (\mathcal{L}_\xi^0 - \lambda)F^0(\xi) - (\mathcal{L}_\xi^0 D^0)\lambda - D^0(\mathcal{L}_\xi^0\lambda) + \lambda D^0\lambda \\ &= \mathcal{L}_\xi^0 F^0(\xi) - D^0(D_\xi^0\lambda - \frac{1}{2}\lambda^2) \end{aligned}$$

since $\mathcal{L}_\xi^0 D^0 = -F^0(\xi)$ on L^{-1} . □

There is still an obstruction to solving this, since it implies $-2F^0(\xi)\rho = \mathcal{L}_\xi^0 \mathcal{L}_\xi^0 F^0(\xi)$.

6.3. Theorem VI. *Let (M, c) be a selfdual conformal 4-manifold with a selfdual projective conformal submersion $\pi: M \rightarrow B$ over an Einstein-Weyl space (B, c_B, D^B) . Then with respect to an arbitrary projective coordinate t on $M \rightarrow B$,*

the conformal structure on M is

$$\mathbf{c} = \pi^* \mathbf{c}_B + (t^2 w_2 + t w_1 + w_0)^{-2} (dt + t^2 A_2 + t A_1 + A_0)^2,$$

where

$$*_B(D^B w_2 + A_2 w_1 - A_1 w_2) = dA_2 + A_2 \wedge A_1,$$

$$*_B\left(\frac{1}{2}D^B w_1 + A_2 w_0 - A_0 w_2\right) = \frac{1}{2}dA_1 + A_2 \wedge A_0,$$

and

$$*_B(D^B w_0 + A_1 w_0 - A_0 w_1) = dA_0 + A_1 \wedge A_0,$$

for some $w_0, w_1, w_2 \in C^\infty(B, L_B^{-1})$ and $A_0, A_1, A_2 \in C^\infty(B, T^*B)$. Conversely for any solution of these projective monopole equations on an Einstein-Weyl space B , the conformal structure given by the above formula is selfdual, and the above decomposition defines a selfdual projective conformal submersion over B .

Proof. As in the proof of Theorem IV, this amounts to computing the equations of Proposition 4.9, now with $w = t^2 w_2 + t w_1 + w_0$ and $A = t^2 A_2 + t A_1 + A_0$. This leads to the quadratic expressions

$$\begin{aligned} D^B w + \dot{A}w - A\dot{w} &= t^2(D^B w_2 + A_2 w_1 - A_1 w_2) \\ &\quad + 2t\left(\frac{1}{2}D^B w_1 + A_2 w_0 - A_0 w_2\right) + D^B w_0 + A_1 w_0 - A_0 w_1 \\ dA + \dot{A} \wedge A &= t^2(dA_2 + A_2 \wedge A_1) + 2t\left(\frac{1}{2}dA_1 + A_2 \wedge A_0\right) + dA_0 + A_1 \wedge A_0 \end{aligned}$$

and equating coefficients (in t) of the resulting equations completes the proof. \square

Note that $D_\xi^g = D_\xi^0 + 2t w_2 + w_1$ and $D^{t\mu_g} = D^0 + (t w_2 - t^{-1} w_0)\xi - t A_2 + t^{-1} A_0$. The projective structure $(D_\xi^0)^2 + \rho$ is given by $\rho = w_0 w_2 - \frac{1}{4} w_1^2$.

The equations arising in this theorem may be identified as $\mathrm{SL}(2, \mathbb{R})$ Einstein-Weyl Bogomolny equations: writing

$$w = \begin{pmatrix} \frac{1}{2} w_1 & w_0 \\ -w_2 & -\frac{1}{2} w_1 \end{pmatrix}, \quad A = \begin{pmatrix} \frac{1}{2} A_1 & A_0 \\ -A_2 & -\frac{1}{2} A_1 \end{pmatrix},$$

yields an $\mathfrak{sl}(2, \mathbb{R})$ -valued density and connection 1-form. The equations of the above Theorem now become $*_B(D^B + adA)w = F^A := dA + A \wedge A$.

7. TWISTOR THEORY OF CONFORMAL SUBMERSIONS

The constructions discussed so far have a natural interpretation on the *twistor space* Z of M . This is a complex manifold fibering over M whose fibres are the antiselfdual complex structures on each tangent space of M (see [1, 2]). The antipodal map on each fibre defines a real structure (antiholomorphic involution) σ on Z , so the fibres of Z are real, i.e., σ -invariant. Each fibre Z_x has normal bundle $N^{(x)} \cong \mathcal{O}(1) \oplus \mathcal{O}(1)$ and so the fibres are precisely the real lines amongst the “twistor lines”, which are the holomorphic deformations of a typical fibre. Hence real holomorphic sections of $N^{(x)}$ over Z_x are constant as maps from Z_x to $T_x M$.

Each Weyl derivative D induces a connection on $\pi_Z: Z \rightarrow M$ and hence a projection $v^D: TZ \rightarrow VZ$ onto the vertical bundle of Z . Under a change of Weyl derivative, $v^{D+\gamma}(U) = v^D(U) + \llbracket \gamma \Delta d\pi_Z(U), J \rrbracket$ for $U \in T_J Z$. If K is any vector field on M and $J \in Z_x$, then the commutator $\llbracket DK - \frac{1}{2} \mathcal{L}_K \mathbf{c}, J \rrbracket$ is a skew endomorphism of $T_x M$ anticommuting with J (since $\mathcal{L}_K \mathbf{c} = 2 \mathrm{sym}_0 DK$) and hence an element of

$V_J Z = T_J(Z_x)$. The lift $K^{\mathbb{C}}$ of K to Z defined by $v^D(K^{\mathbb{C}}) = \llbracket DK - \frac{1}{2}\mathcal{L}_K \mathbb{C}, J \rrbracket$ is easily seen to be independent of the choice of D , and is a holomorphic vector field iff K is a conformal vector field, in which case $v^D(K^{\mathbb{C}}) = \llbracket DK, J \rrbracket$ (see [10]). This generalizes to congruences.

7.1. Proposition. *Congruences ξ on M (up to a sign) are in bijective correspondence with complex line subbundles L^ξ of TZ which are σ -invariant and transverse to the real twistor lines.*

L^ξ is a holomorphic subbundle of TZ iff ξ is a selfdual conformal submersion. The holomorphic structure on this line bundle corresponds, under the Ward correspondence, to the minimal Weyl derivative D^0 .

Sketch proof. Given ξ , let L^ξ be the complex span of those vectors U in $T_J Z$ with $d\pi^Z(U) = \mu\xi$ and $v^D(U) = \mu\llbracket(D^0 \otimes D)\xi - \frac{1}{2}\mathcal{L}_\xi^0 \mathbb{C}, J\rrbracket$ for some element μ of $L_{\pi(J)}^1$ (recall that $\frac{1}{2}\mathcal{L}_\xi^0 \mathbb{C} = \text{sym}_0(D^0 \otimes D)\xi$). This line subbundle L^ξ of TZ naturally isomorphic to $\pi^* L^1 \otimes \mathbb{C}$ and is clearly transverse to the real twistor lines. Conversely, a σ -invariant complex line subbundle L^ξ transverse to a real twistor line Z_x must be degree 0 and the real sections define a congruence ξ up to sign, and identify L^ξ with $\pi^* L^1 \otimes \mathbb{C}$.

By the Ward correspondence, a $\bar{\partial}$ -operator on L^ξ corresponds to a Weyl derivative D^0 on M , and L^ξ holomorphic if D^0 is selfdual [10]. If so, the inclusion of L^ξ into TZ may be viewed as a section $\xi^{\mathbb{C}}$ of the holomorphic bundle $(L^\xi)^{-1} \otimes TZ$ and one finds that $\xi^{\mathbb{C}}$ is holomorphic iff $\mathcal{L}_\xi^0 \mathbb{C} = 0$. \square

This gives a twistorial explanation for the theorems of section 4.

1. The distribution L^ξ on Z given by a selfdual conformal submersion ξ integrates a holomorphic foliation with one dimensional leaves. Since L^ξ is trivial on each twistor line, the twistor lines map to rational curves (called “minitwistor lines”) with normal bundle $\mathcal{O}(2)$ in the (local) quotient space \mathcal{S} , which is the “minitwistor space” that gives rise to the Einstein-Weyl structure on M/ξ [13].
2. The condition that ξ is holomorphic with respect to an antiselfdual complex structure J is simply the condition that the image of J (as a section of Z) is a union of leaves of the foliation determined by ξ . This divisor \mathcal{D} in Z therefore descends to a divisor \mathcal{C} in the minitwistor space, which in turn determines a shear-free geodesic congruence [7]. The correspondence between hypercomplex and hyperCR spaces follows from the fact that $[\mathcal{D} - \bar{\mathcal{D}}]$ is trivial if $[\mathcal{C} - \bar{\mathcal{C}}]$ is trivial: if so this gives a map from \mathcal{S} to $\mathbb{C}P^1$ and hence from Z to $\mathbb{C}P^1$. On the other hand, the correspondence between scalar-flat Kähler metrics and Toda Einstein-Weyl spaces has a more subtle generalization because the canonical bundle K_Z is no longer the pullback of $K_{\mathcal{S}}$ and so $[\mathcal{D} - \bar{\mathcal{D}}]K_Z^{1/2}$ is not the pullback of $[\mathcal{C} - \bar{\mathcal{C}}]K_{\mathcal{S}}^{1/2}$.
3. Compatible Einstein-Weyl structures D on selfdual conformal manifolds are Einstein or locally hypercomplex [3, 21] and correspond to holomorphic rank two distributions $H^D = \ker v^D$ on Z . The twisted 1-form $\theta \in H^0(Z, (L^D)^{-1}K_Z^{-1/2}T^*Z)$ defining H^D can be contracted with $\xi^{\mathbb{C}} \in H^0(Z, (L^\xi)^{-1}TZ)$ to give $\theta(\xi^{\mathbb{C}})$, a holomorphic section of $(L^D)^{-1}(L^\xi)^{-1}K_Z^{-1/2}$, which has degree two on each twistor line.

If this section is not identically zero, then the corresponding divisor gives rise to a complex structure. On the other hand, if the section is identically zero, then L^ξ is a subbundle of H^D and so $[(D^0 \otimes D)\xi, J] = 0$ for all antiselfdual almost complex structures J .

The affine and projective cases may also be interpreted twistorially: the holomorphic bundle $Z \rightarrow \mathcal{S}$ is an affine or projective line bundle. Such bundles arise as affine subspaces or projectivizations of rank two vector bundles trivial on minitwistor lines. Since these correspond to Einstein-Weyl monopoles via a generalized Hitchin-Ward correspondence [14], it is no surprise that the equations are affine and projective monopole equations.

When the Einstein-Weyl structure is Einstein with nonzero scalar curvature, H^D is a contact distribution, whereas when it is locally hypercomplex, H^D is integrable. Therefore, if the skew symmetric part of $(D^0 \otimes D)\xi$ is selfdual (i.e., if L^ξ is a subbundle of H^D), I will say that ξ is *Legendrian* or *triholomorphic* in the case that the scalar curvature is nonzero or zero respectively.

In the Legendrian case, the leaves of the foliation of Z given by ξ are Legendrian curves in a contact manifold. Let \mathcal{S} be the (local) quotient and let $Z^{(y)}$ be the leaf corresponding to a point $y \in \mathcal{S}$. Then at each point J of $Z^{(y)}$, the contact distribution projects onto a one dimensional subbundle of $T_y\mathcal{S}$, giving a holomorphic map of complex curves from $Z^{(y)}$ to $P(T_y\mathcal{S})$. This map cannot be constant, as the contact distribution is non-integrable, so it is locally an isomorphism. By its very definition, this isomorphism identifies the contact distribution on Z with the canonical contact distribution on $P(T\mathcal{S}) \cong P(T^*\mathcal{S})$, where a line in $T_y\mathcal{S}$ is identified with its annihilator in $T_y^*\mathcal{S}$. Therefore, Z can be locally identified (in a neighbourhood of any twistor line) with the projectivized cotangent bundle of \mathcal{S} . This is Hitchin's construction of the selfdual Einstein metric (with nonzero scalar curvature) "filling in" a 3-dimensional Einstein-Weyl space [13]. LeBrun [16] has given such a construction for any real analytic conformal 3-manifold B , and Hitchin observes that the choice of a compatible Einstein-Weyl structure on B (if one exists) equips the selfdual Einstein metric with a conformal submersion onto B . The discussion here *characterizes* the conformal submersions arising in this way.

7.2. Theorem VII. *Let M be a selfdual Einstein manifold with nonzero scalar curvature. Then M arises from Hitchin's construction iff it admits a Legendrian selfdual conformal submersion.*

The scalar curvature is usually taken to be negative: it is in such a real slice that the original conformal 3-manifold appears as a conformal infinity.

When the Einstein-Weyl space is hyperCR, its minitwistor space fibres over \mathbb{CP}^1 and the vertical bundle of this fibration is transverse to the minitwistor lines. This line subbundle of $T\mathcal{S}$ defines a section X of $P(T\mathcal{S})$ which does not intersect the lifted minitwistor lines, and hence does not intersect nearby twistor lines. On removing this section $P(T\mathcal{S}) \setminus X(\mathcal{S})$ is an affine bundle over \mathcal{S} (and is still a twistor space). Hence one can expect to carry out the Hitchin-LeBrun construction explicitly in this case, using the affine monopole equations.

In general, $P(T\mathcal{S}) = P(K_S^{1/2}T\mathcal{S})$ is at least a projective line bundle, and so, corresponding to $K_S^{1/2}T\mathcal{S}$, which is trivial on minitwistor lines and has trivial determinant, there should be a canonical solution of the $\mathrm{SL}(2, \mathbb{R})$ Einstein-Weyl Bogomolny equations on any Einstein-Weyl space, yielding a general formula for the Hitchin-LeBrun construction on any Einstein-Weyl space. In the final section I shall find this canonical solution directly.

8. EINSTEIN-WEYL STRUCTURES AND CONFORMAL SUBMERSIONS

If g is a Riemannian metric and $D = D^g + \omega$ is Einstein-Weyl, then it is well known that $D^g - \omega$ is Einstein-Weyl if and only if ω is dual, with respect to g , to a conformal vector field. If ω is also divergence-free with respect to g , i.e., g is a *Gauduchon gauge* for D , then ω is dual to a Killing field of g . (See e.g. [6, 7, 10, 21] for more information on Einstein-Weyl geometry.)

8.1. Proposition. *Suppose that (M, c) is a conformal manifold and (D^+, D^-) are compatible Einstein-Weyl structures on M . Define a 1-form $\theta := \frac{1}{2}(D^+ - D^-)$ and a Weyl derivative $D := \frac{1}{2}(D^+ + D^-)$ (the barycentre) so that $D^\pm = D \pm \theta$. Then on the open set where θ is nonvanishing, $\xi = \theta/|\theta|$ is a conformal submersion with minimal Weyl derivative $D^0 = 2D - D^{|\theta|} = D - |\theta|^{-1}D|\theta|$.*

(Here $D^{|\theta|}$ is the exact Weyl derivative defined by $D^{|\theta|}|\theta| = 0$. Since $|\theta|$ is a section of L^{-1} , this means that $D^{|\theta|} = D + |\theta|^{-1}D|\theta|$.)

Proof. The standard formula for the dependence of the (normalized) Ricci tensor on the Weyl derivative gives

$$r^{D^\pm} = r^D \mp D\theta + \theta \otimes \theta - \frac{1}{2}|\theta|^2 \mathrm{id}$$

and hence $D\theta = \frac{1}{2}(r^{D^-} - r^{D^+})$. Since D^\pm are both Einstein-Weyl, $\mathrm{sym}_0 D\theta = 0$ and so one can write $D\theta = \sigma \mathrm{id} + F$ where σ is a section of L^{-2} and F is a skew endomorphism of weight -2 . Direct calculations give

$$D^0 = D - |\theta|^{-1}(\sigma\xi - F(\xi))$$

$$D^{|\theta|} = D + |\theta|^{-1}(\sigma\xi - F(\xi)).$$

One now readily checks that ξ is a conformal submersion. □

The following diagram in the affine space of Weyl derivatives summarizes this Proposition, where $\gamma = |\theta|^{-1}D|\theta| = |\theta|^{-1}(\sigma\xi - F(\xi))$.

$$\begin{array}{ccccc} & & D^+ & & \\ & \nearrow & \uparrow \theta & \searrow & \\ D^0 & \xrightarrow{\gamma} & D & \xrightarrow{\gamma} & D^{|\theta|} \\ & \searrow & \uparrow \theta & \nearrow & \\ & & D^- & & \end{array}$$

In particular D^0 is exact iff D is exact. This holds, for instance if D^\pm are both Levi-Civita derivatives of Einstein metrics.

I now specialize to four dimensions and Einstein-Weyl structures with selfdual Faraday curvature. On a selfdual conformal 4-manifold, Einstein-Weyl structures necessarily have selfdual Faraday curvature [3] and are either Einstein or locally hypercomplex [21].

8.2. Theorem VIII. *Let D^\pm be Einstein-Weyl structures whose Faraday curvatures F^{D^\pm} are selfdual. Then, with the notation of the previous proposition, ξ is a selfdual conformal submersion, and also $\text{alt}(D^0 \otimes D^{sd})\xi$ is selfdual.*

Proof. Since F^{D^+} and F^{D^-} are both selfdual, so is F^D , and hence so is F^0 , since $D^{|\theta|}$ is exact. This means that ξ is a selfdual conformal submersion. Furthermore F is selfdual, since it is equal to $\frac{1}{4}(F^{D^+} - F^{D^-})$. Consequently $\omega = -2|\theta|^{-1}F(\xi)$ and so $D = D^{sd} + |\theta|^{-1}\sigma\xi$. Therefore D^\pm both differ from D^{sd} by a vertical 1-form, and so the antiselfdual part of $(D^0 \otimes D^\pm)\xi$ vanishes. \square

In the language of the previous section, the final condition of ξ means that ξ is Legendrian if D^\pm is the Levi-Civita derivative of an Einstein metric, and triholomorphic if D^\pm is the Obata derivative of a hypercomplex structure.

Another picture in the affine space of Weyl derivatives may be helpful.

$$\begin{array}{ccccc}
 D^0 & \xrightarrow{\frac{1}{2}\omega} & D^{sd} & \xrightarrow{\frac{1}{2}\omega} & D^B \\
 & \searrow \gamma & \downarrow |\theta|^{-1}\sigma\xi & & \downarrow 2|\theta|^{-1}\sigma\xi \\
 & & D & \searrow \gamma & D^{|\theta|}
 \end{array}$$

When both Einstein-Weyl structures are (locally) hypercomplex $\sigma = 0$ and so $D^B = D^{|\theta|}$ and the Einstein-Weyl structure on B is Einstein. Indeed, since the Einstein-Weyl quotient B admits two hyperCR structures, it must be the round 3-sphere metric [11]. Such bi-hypercomplex structures in four dimensions have been studied already by Apostolov and Gauduchon [private communication] and they have an elegant construction of the conformal submersion that arises in this case, which I shall briefly describe.

If the hypercomplex structures corresponding to D^+ and D^- are (I_1^+, I_2^+, I_3^+) and (I_1^-, I_2^-, I_3^-) , then since both give oriented orthonormal frames for $L^2\Lambda_-^2 T^*M$, they are related by an $\text{SO}(3)$ -valued function: $I_i^+ = A_{ij}I_j^-$. Applying D to this equation gives $dA_{ij}(X)I_j^- = -2[\theta \triangle X, A_{ij}I_j^-]$, which implies, after some manipulations, that $dA_{ij}(X) = -A_{ik}\varepsilon_{jkl}\theta(I_\ell^- X)$. In particular, $dA_{ij}(\xi) = 0$ so this map to $\text{SO}(3)$ factors through the conformal submersion ξ . To see that the map to $\text{SO}(3)$ actually is the conformal submersion ξ , one computes $dA_{ij}(X)dA_{ij}(X) = 2(|\theta|^2|X|^2 - \theta(X)^2)$.

Conversely, by Theorem II, any selfdual conformal submersion over the round 3-sphere metric is bi-triholomorphic with respect to a bi-hypercomplex structure. The generalized monopole equations of Proposition 4.9 in the gauge $w = |\theta| = \mu_{S^3}^{-1}$ reduce to $*_B \dot{A}w = dA + \dot{A} \wedge A$, which have been obtained independently by Belgun and Moroianu.

9. SELFDUAL EINSTEIN METRICS AND HYPERCOMPLEX STRUCTURES

In this section I study selfdual conformal 4-manifolds M admitting a compatible Einstein metric with nonzero scalar curvature and a compatible hypercomplex structure. All such structures are obtained by applying the Hitchin-LeBrun construction to a hyperCR Einstein-Weyl space B and the Einstein metric will be found explicitly in terms of the Einstein-Weyl structure on B and a special solution of the affine monopole equations.

The work of the previous section shows that if $D^g = D - \theta$ is the Levi-Civita derivative of the Einstein metric and $D^{Ob} = D + \theta$ is the Obata derivative of the hypercomplex structure, then $\xi = \theta/|\theta|$ is a Legendrian triholomorphic selfdual conformal submersion. The main goal in this section is to add one more adjective to this list and prove that the submersion is affine.

To do this, a section w_1 of L^{-1} must be found with $D^0 w_1 = F^0(\xi)$: w_1 is then linear part of the affine monopole w and the affine structure is given by $D_\xi^0 + w_1$.

9.1. Proposition. *Write $D^{Ob} = D^{sd} - \kappa\xi$. Then $D^0(2\kappa) = F^0(\xi)$.*

Proof. First recall that κ is basic, i.e., $D_\xi^0 \kappa = 0$. Now D^0 and D^{Ob} are gauge equivalent, since D^g and $D^{|\theta|}$ are both exact. This implies that

$$0 = d(\omega - 2\kappa\xi)(\xi) = d\omega(\xi) + D^0(2\kappa) = F^B(\xi) - F^0(\xi) + D^0(2\kappa)$$

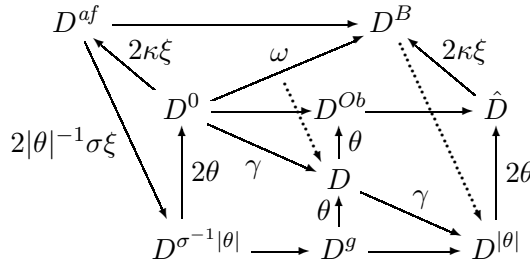
which proves the proposition since $F^B(\xi) = 0$. \square

This shows that the linear part of the affine monopole is twice the κ monopole of the hyperCR space $B = M/\xi$, and so it satisfies $*_B D^B(2\kappa) = F^B$ [11, 7]. Fix a gauge on B so that the Einstein-Weyl structure is given by a metric g_B and a 1-form ω_B . Then $w_1 = 2\kappa$ and one can take $A_1 = \omega_B$. In order to fix the affine gauge on M note that σ is a nonzero constant multiple of the scalar curvature of g and so $D^g \sigma = 0$ since g is Einstein. In particular σ is nonvanishing by assumption.

9.2. Proposition. *Define $D^{af} = D^0 + 2\kappa\xi$. Then $D_\xi^{af}(\sigma^{-1}|\theta|) = -2$ so $\sigma^{-1}|\theta|$ is an affine section of L^1 with respect to the affine structure.*

Proof. Observe that $(D^0 - 2\theta)(\sigma^{-1}|\theta|) = 0$, i.e., $D^{\sigma^{-1}|\theta|} = D^0 - 2\theta$ and so $D^{af} = D^{\sigma^{-1}|\theta|} + 2\theta + 2\kappa\xi = D^{\sigma^{-1}|\theta|} - 2|\theta|^{-1}\sigma\xi$. Hence $D^{af}(\sigma^{-1}|\theta|) = -2\xi$. \square

One more diagram in the affine space of Weyl derivatives may again clarify the situation, where the Weyl derivative \hat{D} with $\hat{D}(|\theta|^{-1}) = 2\xi$ has been introduced for completeness.



It follows from the above proposition that the affine coordinate t can be taken to be a multiple of $\sigma^{-1}|\theta|$ and so $|\theta|^{-1}\sigma\xi$ is a multiple of $t^{-1}(dt + tA_1 + A_0)$. Now note that $D^B = D^{|\theta|} - 2|\theta|^{-1}\sigma\xi$ and so $-2d(|\theta|^{-1}\sigma\xi) = F^B$, $A_0 = 0$ and $|\theta|^{-1}\sigma\xi = -\frac{1}{2}(t^{-1}dt + \omega_B)$. The remaining gauge freedom can be fixed by taking $w_0 = \mu_B^{-1}$.

Certainly $(1 + 2t\mu_B\kappa, t\omega_B)$ is a solution of the affine monopole equations on a hyperCR Einstein-Weyl space, yielding a selfdual space with a hypercomplex structure by Theorem II. The Einstein gauge is the barycentre of the $|\theta|^{-1}$ gauge and the $\sigma^{-1}|\theta|$ gauge (both of which have been identified in terms of the affine structure) and one easily checks that this encodes the Einstein equation, since D^{Ob} is Einstein-Weyl and $\text{sym}_0 D\theta = 0$. Hence, writing $\mu_t = t\mu_B$, the following theorem is obtained.

9.3. Theorem IX. *Let (c_B, D^B) be a 3-dimensional hyperCR Einstein-Weyl space with twist κ . Then the metric*

$$g = \frac{1}{\mu_t^2}((1 + 2\mu_t\kappa)\pi^*c_B + (1 + 2\mu_t\kappa)^{-1}(D^B\mu_t)^2)$$

on $\pi: M \rightarrow B$ is selfdual Einstein with nonzero scalar curvature and admits a compatible hypercomplex structure, where μ_t is a section of L^1 increasing along the fibres. Any selfdual Einstein metric with a compatible hypercomplex structure arises in this way.

Note that c_B arises as a conformal infinity at $\mu_t = 0$.

Explicit selfdual Einstein metrics can be found by applying this construction to explicit hyperCR Einstein-Weyl spaces. To the best of my knowledge, the known examples are the Einstein-Weyl spaces with geodesic symmetry, and the hyperCR-Toda spaces. In the former case note that the twist of the hyperCR structure is *minus* the twist of the geodesic symmetry and so the examples of section 5.2 (with $h = f$) are reobtained [8]—these are the Pedersen metrics [20] when h is constant.

On the other hand, the hyperCR-Toda spaces yield new selfdual Einstein metrics with compatible hypercomplex structures and no continuous symmetries:

$$g = \frac{1}{t^2}(H(|z + h|^2 g_{S^2} + dz^2) + H^{-1}(dt + t\omega_B)^2)$$

where
$$H = 1 + \frac{i(h - \bar{h})t}{|z + h|^2}, \quad \omega_B = -\frac{2z + h + \bar{h}}{|z + h|^2}dz$$

and h is holomorphic on an open subset of S^2 .

HyperCR Einstein-Weyl spaces are not well understood: the results of this section perhaps provide motivation for further investigations. Alternatively, one may study hypercomplex selfdual Einstein 4-manifolds directly. Along these lines Apostolov and Gauduchon have investigated compatible selfdual complex structures and obtained a nice characterization of the Pedersen metrics.

10. THE HITCHIN-LEBRUN CONSTRUCTION

There is an interesting gauge transformation one can apply to the metric of Theorem IX: on replacing μ_t by the new projective coordinate $\mu_t/(1 - \mu_t\kappa)$, the

metric becomes:

$$g = \frac{1}{\mu_t^2} \left((1 - \mu_t^2 \kappa^2) \pi^* \mathbf{c}_B + (1 - \mu_t^2 \kappa^2)^{-1} (D^B \mu_t + \mu_t^2 D^B \kappa) \right).$$

Now on a hyperCR Einstein-Weyl space, $D^B \kappa = -\frac{1}{2} *_B F^B$ and $\kappa^2 = \frac{1}{6} \text{scal}^B$ (see [11]), so this form of the metric makes sense for any Einstein-Weyl space. In this final section, I prove the following theorem.

10.1. Theorem X. *Let (\mathbf{c}_B, D^B) be an arbitrary 3-dimensional Einstein-Weyl structure with Faraday curvature F^B and scalar curvature scal^B . Then*

$$g = \left(1 - \frac{1}{6} \mu_t^2 \text{scal}^B\right) \mu_t^{-2} \pi^* \mathbf{c}_B + \left(1 - \frac{1}{6} \mu_t^2 \text{scal}^B\right)^{-1} \left(\mu_t^{-1} D^B \mu_t - \frac{1}{2} \mu_t *_B F^B\right)^2$$

is a selfdual Einstein metric of nonzero scalar curvature, with a Legendrian selfdual conformal submersion over B . Here μ_t is a section of L^1 increasing along the fibres of the conformal submersion $\pi: M \rightarrow B$ and the conformal structure \mathbf{c}_B is the conformal infinity at $\mu_t = 0$. Any such selfdual Einstein metric arises in this way.

Strictly speaking, the above metric is only positive definite for $\mu_t^2 \text{scal}^B < 1$ and in this region the scalar curvature is negative. For $\mu_t^2 \text{scal}^B > 1$ the negation of the above metric is positive definite and has positive scalar curvature.

Note that the Einstein metric can be written in a gauge μ_B by writing $\mu_t = t\mu_B$, $g_B = \mu_B^{-2} \mathbf{c}_B$ and $D^B = D^{\mu_B} + \omega_B$. Then

$$g = \frac{1}{t^2} \left(\left(1 - \frac{1}{6} t^2 \mu_B^2 \text{scal}^B\right) g_B + \left(1 - \frac{1}{6} t^2 \mu_B^2 \text{scal}^B\right)^{-1} (dt + t\omega_B - \frac{1}{2} t^2 \mu_B *_B F^B)^2 \right).$$

The theorem is proven using the projective monopole equations. The conformal structure is determined by a canonical $\text{SL}(2, \mathbb{R})$ monopole on $L_B^{1/2} \oplus L_B^{-1/2}$. In a gauge μ_B , the Higgs field and connection 1-form are given by:

$$w = \begin{pmatrix} 0 & \mu_B^{-1} \\ \frac{1}{6} \mu_B \text{scal}^B & 0 \end{pmatrix}, \quad A = \begin{pmatrix} \frac{1}{2} \omega_B & 0 \\ \frac{1}{2} \mu_B *_B F^B & -\frac{1}{2} \omega_B \end{pmatrix}.$$

The connection is therefore $D^B + *_B F^B$, with $*_B F^B$ acting from $L_B^{1/2}$ to $L_B^{-1/2}$, while the Higgs field is $1 + \text{scal}^B$ in $L_B^{-1} \otimes (\text{Hom}(L_B^{-1/2}, L_B^{1/2}) \oplus \text{Hom}(L_B^{1/2}, L_B^{-1/2}))$.

The Einstein-Weyl Bogomolny equations are:

$$\begin{aligned} -\frac{1}{6} *_B (D^B (\mu_B \text{scal}^B) - \omega_B \mu_B \text{scal}^B) &= -\frac{1}{2} d(\mu_B *_B F^B) + \frac{1}{2} \omega_B \wedge \mu_B *_B F^B, \\ \frac{1}{2} F^B &= \frac{1}{2} d\omega_B, \end{aligned}$$

$$D^B (\mu_B^{-1}) + \omega_B \mu_B^{-1} = 0.$$

Since $D^B \mu_B = \omega_B \mu_B$ and $F^B = d\omega_B$, the only nontrivial equation is the first one, which reduces to $\frac{1}{6} D^B \text{scal}^B = -\frac{1}{2} *_B d *_B F^B = \frac{1}{2} \delta^B F^B$, where δ^B is the twisted exterior divergence. This equation is satisfied automatically: it is the differential Bianchi identity for the Weyl connection in Einstein-Weyl geometry [10, 6].

It follows from Theorem VI that the metric g is selfdual with a selfdual projective conformal submersion. It remains, therefore, to prove that g is Einstein and ξ is Legendrian. To do this, observe that the Levi-Civita derivative of g is the barycentre

of the Levi-Civita derivatives of μ_t and tw^{-1} . Simple calculations show that these are given by

$$D^B - \mu_t^{-1} D^B \mu_t = D^B - (1 - \frac{1}{6} \mu_t^2 \text{scal}^B) \mu_t^{-1} \xi - \frac{1}{2} \mu_t *_B F^B$$

and

$$D^w - t^{-1} dt = D^0 - (1 + \frac{1}{6} \mu_t^2 \text{scal}^B) \mu_t^{-1} \xi + \frac{1}{2} \mu_t *_B F^B.$$

Hence $D^g = D^{sd} - \mu_t^{-1} \xi$ (which will give the Legendrian property) and consequently,

$$r^g = r^{sd} + D^B(\mu_t^{-1}) \otimes \xi + \mu_t^{-1} (D^0 \otimes D^{sd}) \xi + \mu_t^{-2} (\xi \otimes \xi - \frac{1}{2} id),$$

where I have written $D^{sd} = D^B \otimes D^0 \otimes D^{sd}$ on $T^*M = L^{-1} \otimes L^{-1} \otimes TM$. This simplifies to give

$$r^g = r^{sd} - \frac{1}{2} *_B F^B \otimes \xi + \frac{1}{6} \text{scal}^B \xi \otimes \xi - \frac{1}{2} \mu_t^{-2} id + \mu_t^{-1} (D^0 \otimes D^{sd}) \xi.$$

Finally substituting from Proposition 4.7 yields $\text{sym } r^g = \frac{1}{2} (\frac{1}{6} \text{scal}^B - \mu_t^{-2}) id$, and so g is Einstein, with scalar curvature $-12 \mu_t^{-2} (1 - \frac{1}{6} \text{scal}^B \mu_t^2)$. This completes the proof of Theorem X.

Examples arising from hyperCR Einstein-Weyl spaces have already been discussed. One source of further examples are the Ward-Toda spaces [24, 5]

$$g = (V_\rho^2 + V_\eta^2)(d\rho^2 + d\eta^2) + d\psi^2$$

$$\omega = \frac{2V_\rho V_\eta d\eta + (V_\rho^2 - V_\eta^2)d\rho}{\rho(V_\rho^2 + V_\eta^2)}$$

where V is an axially symmetric harmonic function: $(\rho V_\rho)_\rho + \rho V_{\eta\eta} = 0$. These spaces admit a symmetry generated by $\partial/\partial\psi$ and hence so will their Hitchin-LeBrun metrics. Already in these examples, the Faraday and scalar curvatures are quite formidable, so these Einstein metrics are not at all simple. Nevertheless, they can be made completely explicit, and will undoubtedly repay further study.

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